Math 487
Exam 3 - Practice Problem Solutions

1. Consider the following functions:

- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + x$
- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$
- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x - 3$
- $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (x, y^2)$
- $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (x^3, y)$
- $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (5x, 2y)$
- $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g(x, y) = (-x, -y)$

(a) Which of these mappings are transformations? Justify your answer.

- Consider $f(x) = x^2 + x$. Since $f(0) = 0$ and $f(-1) = 1 - 1 = 0$, f is not 1-1 and hence is not a transformation.
- Consider $f(x) = 2x$. First suppose $f(a) = f(b)$. Then $2a = 2b$, so $a = b$ hence f is 1-1.

  Next, let $r \in \mathbb{R}$. Let $a = \frac{r}{2}$. Then $f(a) = 2 \cdot \frac{r}{2} = r$. Hence f is also onto, hence f is a transformation.
- Consider $f(x) = x - 3$. First suppose $f(a) = f(b)$. Then $a - 3 = b - 3$, so $a = b$ hence f is 1-1.

  Next, let $r \in \mathbb{R}$. Let $a = r+3$. Then $f(a) = (r+3)-3 = r$. Hence f is also onto, hence f is a transformation.
- Consider $g(x, y) = (x, y^2)$. Notice that $g(0, 1) = (0, 1)$ and $g(0, -1) = (0, 1)$. Therefore, g is not 1-1 and hence is not a transformation.
- Consider $g(x, y) = (x^3, y)$. First suppose $g((a, b)) = g((c, d))$. Then $a^3 = c^3$, so $a = c$ and $b = d$. hence g is 1-1.

  Next, let $(r, s) \in \mathbb{R}$. Let $a = \sqrt[3]{r}$ and $b = s$. Then $g((a, b)) = (r, s)$. Hence g is also onto, hence g is a transformation.
- Consider $g(x, y) = (5x, 2y)$. First suppose $g((a, b)) = g((c, d))$. Then $5a = 5c$, so $a = c$ and $2b = 2d$, so $b = d$. hence g is 1-1.

  Next, let $(r, s) \in \mathbb{R}$. Let $a = \frac{r}{5}$ and $b = \frac{s}{2}$. Then $g((a, b)) = (r, s)$. Hence g is also onto, hence g is a transformation.
- Consider $g(x, y) = (-x, -y)$. First suppose $g((a, b)) = g((c, d))$. Then $-a = -c$, so $a = c$ and $-b = -d$, so $b = d$. hence g is 1-1.

  Next, let $(r, s) \in \mathbb{R}$. Let $a = -r$ and $b = -s$. Then $g((a, b)) = (r, s)$. Hence g is also onto, hence g is a transformation.

(b) Which of these mappings are affine transformations? Justify your answer.

From above, $f(x) = x^2 + x$ and $g(x, y) = (x, y^2)$ are not transformations, so they are also not affine transformations.

Consider $g(x, y) = (x^3, y)$. Notice that $(0, 0)$, $(1, 1)$, $(2, 2)$ are collinear since they are all on the line $y = x$. However, $g((0, 0)) = (0, 0)$, $g((1, 1)) = (1, 1)$, and $g((2, 2)) = (8, 2)$. This first two points are both on the line $y = x$ while the third point is not on $y = x$. Therefore, this transformation does not preserve lines, hence it cannot be an affine transformation.

It may or may not make sense to call $f(x) = 2x$ and $f(x) = x - 3$ affine, depending on how we define lines since these are not functions of $\mathbb{E}$. Since they both map $\mathbb{R}$ onto $\mathbb{R}$, they preserve the only line in view, so we will consider both of these to be affine.

Consider $g(x, y) = (5x, 2y)$. We first consider how g acts on vertical lines. Given the line $x = a$, points on this line all have the form (a, y). Since $g((a, y)) = (5a, 2y)$, these points are all mapped to the vertical line $x = 5a$. Moreover, this map is onto , since if we set $y = \frac{b}{2}$, we can get $g((a, y)) = (5a, b)$ for any b.

Next, consider a non-vertical line of the form $y = mx + b$. Then $g(a, b) = g(a, ma + b) = (5a, 2ma + 2b)$, and $g(c, d) = g(c, md + b) = (5c, 2mc + 2b)$.

Then the slope of the line between the pair of image points is given by: $m' = \frac{2mc+2b-2ma-2b}{5c-5a} = \frac{2m(c-a)}{5(c-a)} = \frac{2m}{5}$.

Since any pair of inputs are mapped to a pair of points that satisfy this slope equation, this function maps the line into another line. The fact that this map is onto is similar to the argument above.

We will show that $g(x,y) = (-x,-y)$ is affine by showing that it is an isometry below.

(c) Which of these mappings are isometries? Justify your answer.

First notice that all maps that are not affine are not isometries since we know that all isometries preserve lines.

Notice that $f(x) = 2x$ is not an isometry since if $a = 0$ and $b = 1$, $d(0,1) = 1$ while, since $f(0) = 0$ and $f(1) = 2$, $d(f(0), f(1)) = 2$, so distance is not preserved.

$f(x) = x - 3$ is an isometry since given inputs a and b, $d(a,b) = |b-a|$, while $d(f(a), f(b)) = |(b-3)-(a-3)| = |b-a| = d(a,b)$.

Finally, $g(x,y) = (-x,-y)$ is an isometry since $d(g(a,b), g(c,d)) = d((-a,-b),(-c,-d))$
$= \sqrt{(-c-(-a))^2 + (-d-(-b))^2} = \sqrt{(-1)^2(c-a)^2 + (-1)^2(d-b)^2} = \sqrt{(c-a)^2+(d-b)^2} = d((a,b),(c,d))$.

2. Prove each of the following:

(a) The inverse of a transformation is a transformation.

Let f be a transformation from D to R. By definition, f is a function that is both one-to-one and onto. Then we may define the inverse function f^{-1} as follows. For every $r \in R$, since f is 1-1 and onto, there is exactly one $d \in D$ such that $f(d) = r$. Then we define $f^{-1}(r) = d$. Then f^{-1} is a function. Since f is well defined, f^{-1} is onto (for every $d \in D$, there is an $r \in R$ such that $f(r) = d$). Also, f^{-1} is well defined since we for each $r \in R$, there is a $d \in D$ so that $f^{-1}(r) = d$ and there is exactly one such d since f is 1-1. Finally, f^{-1} must be 1-1 since if $f^{-1}(r_1) = f^{-1}(r_2) = d$, $f \circ f^{-1}(r_1) = f \circ f^{-1}(r_2) = f(d)$, so $r_1 = r_2$. Hence f^{-1} is a transformation.

(b) The inverse of an isometry is an isometry.

We already know from above that the inverse of a transformation is a transformation, so we need only prove that the inverse of an isometry preserves the distance between any pair of points. Let f be an isometry from D to R. Let $r_1, r_2 \in R$ and suppose that $d(r_1, r_2) = M$. To obtain a contradiction, suppose that $d(f^{-1}(r_1), f^{-1}(r_2)) = M' \neq M$. Then, since f is an isometry, $M' = d\left(f\left(f^{-1}(r_1)\right), f\left(f^{-1}(r_2)\right)\right) = d(r_1, r_2) = M$. This is a contradiction. Hence f^{-1} is an isometry.

(c) The set of all transformations of a plane forms a group under the operation function composition.

- We showed in HW 3.4 that the composition of two transformations in a transformation. Therefore, the set of transformations of a plane is closed under function composition.
- The identity map is a transformation since it is both 1-1 and onto.
- From part (a) above, the inverse of a transformation is a transformation.
- The fact that function composition is associative is a standard result from the algebra of functions. (i.e. one can show that $f \circ (g \circ h) = (f \circ g) \circ h$).

(d) Given f, an isometry of $\mathbb{E}$ and a triangle $\triangle ABC$, if $f(A) = A'$, $f(B) = B'$, and $f(C) = C'$, then $\triangle ABC \cong \triangle A'B'C'$.

Suppose that $d(A,B) = l_1$, $d(A,C) = l_2$ and $d(B,C) = l_3$. Since f is an isometry, $d(A',B') = d(A,B) = l_1$, $d(A',C') = d(A,C) = l_2$ and $d(B',C') = d(B,C) = l_3$. Then $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$, $\overline{BC} \cong \overline{B'C'}$. Hence, by the SSS Theorem, $\triangle ABC \cong \triangle A'B'C'$.

3. (a) Find homogeneous coordinates for the line $\ell \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}$ containing the points $(-3,1,1)$ and $(1,-2,1)$.

Let $(x_1, x_2, 1)$ be an arbitrary point on the line ℓ containing $(-3,1,1)$ and $(1,-2,1)$. To find ℓ we look at:

$$\begin{vmatrix} x_1 & -3 & 1 \\ x_2 & 1 & -2 \\ 1 & 1 & 1 \end{vmatrix} = x_1(1+2) + 3(x_2+2) + 1(x_2-1) = 0$$

Then $3x_1 + 3x_2 + 6 + x_2 - 1 = 0$ or $3x_1 + 4x_2 + 5 = 0$, so the homogeneous coordinates of ℓ are given by $\begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$.

(b) Find the point of intersection of the lines $\begin{bmatrix} 4 & -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 3 & 2 & -10 \end{bmatrix}$.

To find the point of intersection between the lines $\begin{bmatrix} 4 & -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 3 & 2 & -10 \end{bmatrix}$, we consider the following:

$$\begin{vmatrix} \ell_1 & \ell_2 & \ell_3 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{vmatrix} = \ell_1(10-0) - \ell_2(-40-0) + \ell_3(8+3)$$
$$= 10\ell_1 + 40\ell_2 + 11\ell_3.$$

Thus the point of intersection is: $(\frac{10}{11}, \frac{40}{11}, 1)$.

(c) Find the angle between the lines $\begin{bmatrix} 4 & -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 3 & 2 & -10 \end{bmatrix}$.

To find the angle between the lines $\begin{bmatrix} 4 & -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 3 & 2 & -10 \end{bmatrix}$, recall if we are given $p \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$, and $q \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$, then

$$m(\angle p, q) = \tan^{-1}\left(\frac{p_1q_2 - p_2q_1}{p_1q_1 + p_2q_2}\right) = \tan^{-1}\left(\frac{8+3}{12-2}\right) = \tan^{-1}\left(\frac{11}{10}\right) \approx 47.73°.$$

4. Define each of the following terms:

(a) a transformation of a plane.

A *transformation of a plane* is a transformation that maps points of the plane onto points in the plane.

(b) an isometry.

A transformation which preserves the distance between any pair of points is an *isometry*.

(c) a group.

A nonempty set G is said to form a *group under a binary operation*, $*$, if it satisfies the following conditions:

- If A and B are in G, then $A * B$ is in G. (The set is closed under the operation [closure].)
- There exists an element I in G such that for every element A in G, $I * A = A * I = A$. (The set has an identity.)
- For every element A in G, there is an element B in G such that $A * B = B * A = I$, denoted A^1. (Every element has an inverse.)
- If A, B, and C are in G, then $(A * B) * C = A * (B * C)$. (Associativity)

(d) a reflection.

A *reflection in a line* ℓ is a transformation of a plane, denoted R_ℓ, such that if X is on ℓ, then $R_{ell}(X) = X$, and if X is not on ℓ, then R_ℓ maps X to X' such that ℓ is the perpendicular bisector of $\overline{XX'}$. The line ℓ is called the *axis* of the reflection.

(e) a translation.

A *translation* through a vector $\overrightarrow{PQ}$ is a transformation of a plane, denoted T_{PQ}, such that if T_{PQ} maps X to X', then the vector $\overrightarrow{XX'} = \overrightarrow{PQ}$.

5. (a) Find the matrix for the transformation T_{PQ} given:

   i. $P(2,3,1)$, $Q(5,3,1)$

   Notice that $\overrightarrow{PQ} = \langle 3, 0 \rangle$.

   Therefore, $T_{PQ} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   ii. $P(2,3,1)$, $Q(-1,5,1)$

   Notice that $\overrightarrow{PQ} = \langle -3, 2 \rangle$.

   Therefore, $T_{PQ} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

   (b) Find the matrix for the transformation $R_{C,\theta}$ given:

   i. $C(0,0,1)$ and $\theta = 135°$

   Then $\cos\theta = -\frac{\sqrt{2}}{2}$ and $\sin\theta = \frac{\sqrt{2}}{2}$

   Hence $R_{C,\theta} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   ii. $C(-1,2,1)$ and $\theta = 30°$

   Then $\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$, and $\overrightarrow{OC} = \langle -1, 2 \rangle$

   Notice that $T_{OC} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, $T_{CO} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, and $R_{O,\theta} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   Finally, $R_{C,\theta} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

   We leave it to the reader to complete the matrix multiplication.

   (c) Find the matrix for R_ℓ given:

   i. $\ell \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

   Notice that to map $h \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ onto ℓ, we use rotation about $O(0,0,1)$ with $\theta = 90°$.

   Also notice that $\cos\theta = 0$ and $\sin\theta = 1$. Then $T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   Similarly, rotating about O with $\theta = 270°$ gives T^{-1}, so $T^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The reflection R_h is given by the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Then $R_\ell = TAT^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We leave it to the reader to complete the matrix multiplication.

ii. $\ell \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

Notice that ℓ is the line with slope -1 intersecting the x-axis at the point $C(-1,0,1)$. Therefore, to map $h \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ onto ℓ, we use rotation about $C(-1,0,1)$ with $\theta = 135°$.

Also notice that $\cos\theta = -\frac{\sqrt{2}}{2}$ and $\sin\theta = \frac{\sqrt{2}}{2}$. Then $R_{O,\theta} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $T_{OC} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

and $T_{CO} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Hence $T = R_{C,45°} = T_{OC} R_{O,\theta} T_{CO} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Similarly, rotating about C with $\theta = 225°$ gives T^{-1}, and the reflection R_h is given by the matrix

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Then $R_\ell = TAT^{-1}$.

We leave it to the reader to complete the computations.

(d) Find the matrix for G_{PQ} given $P(2,3,1)$ and $Q(3,3,1)$.

First notice that $\overleftrightarrow{PQ} = \langle 1, 0 \rangle$.

Therefore, $T_{PQ} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Next, notice that since the line $\ell = \overleftrightarrow{PQ}$ is the horizontal line $\begin{bmatrix} 0 & 1 & -3 \end{bmatrix}$, then the translation that takes $h \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ to ℓ is:

$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

Thus $R_\ell = TAT^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

Finally, $G_{PQ} = T_{PQ} TAT^{-1}$. The details of the matrix multiplication are left to the reader.

6. For each statement, determine whether the statement is true or false. Then briefly justify your answer.

(a) Every isometry of $\mathbb{E}$ is a translation, a rotation, or a reflection.

This statement is False. Glide reflections are also isometries of $\mathbb{E}$, and these to not fall into any of the categories listed above.

(b) The product of any two distinct rotations is a translation.

This is False. We often get translations by composing two rotations, but in the case where two rotations about same center point C are composed, we do not get a translation. We get another rotation about C.

(c) A nontrivial translation has no invariant points.

True. To see this, suppose that a translation f has an invariant point P. Then $f(P) = P$. But then $\overline{PP'} = \langle 0, 0\langle$. Since all points are moved along this same vector, the translation is trivial (i.e. it fixes every point in the plane).

(d) A nontrivial rotation has exactly one invariant point.

True. The only point that is fixed by a non-trivial rotation is the center of the rotation.

(e) A nontrivial translation has no invariant lines.

False. Every line that is parallel to the translation vector in standard position at the origin is invariant under a translation.

(f) A nontrivial rotation has no invariant lines.

False. If $\theta = 180$, then any line through the center of the rotation is invariant under the rotation.

(g) A nontrivial reflection has exactly one invariant line.

False. In addition to the axis of reflection, which is clearly invariant, any line that is perpendicular to the axis of reflection is also invariant.

7. Let f be the transformation given by the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

(a) Find the image of the points $P(1, 3, 1)$ and $Q(-2, 5, 1)$ under this transformation.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \end{bmatrix}^T = \begin{bmatrix} 1+0+2 & 0+3+3 & 0+0+1 \end{bmatrix}^T = \begin{bmatrix} 3 & 6 & 1 \end{bmatrix}^T = P'.$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 5 & 1 \end{bmatrix}^T = \begin{bmatrix} -2+0+2 & 0+5+3 & 0+0+1 \end{bmatrix}^T = \begin{bmatrix} 0 & 8 & 1 \end{bmatrix}^T = Q'.$$

(b) Find the image of the line $\begin{bmatrix} 1 & -2 & 4 \end{bmatrix}$ under this transformation.

Let $\begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \end{bmatrix}$ be the image of the line $\begin{bmatrix} 1 & -2 & 4 \end{bmatrix}$ under A.

Then we have the matrix equation: $\begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \end{bmatrix}$

Multiplying, we see that $\ell_1 = 1$, $\ell_2 = -2$, and $2\ell_1 + 3\ell_2 + \ell_3 = 4$, so $2 - 6 + \ell_3 = 4$ or $\ell_3 = 8$.

This the image of $\begin{bmatrix} 1 & -2 & 4 \end{bmatrix}$ under A is $\begin{bmatrix} 1 & -2 & 8 \end{bmatrix}$.

(c) What transformation is this?.

From this form of the matrix A and its action on P, Q, and ℓ we see that it is the translation T_{PQ}, where $\overrightarrow{PQ} = \langle 2, 3\rangle$.

8. Given the points $P(2,1,1)$ and $Q(4,2,1)$

(a) Find the matrix of a *translation* that maps P to Q.

Notice that $\overline{PQ} = \langle 2,1\rangle$. Then the following is a matrix of a translation that takes P to Q:

$$T_{PQ} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Find the matrix of a *reflection* that maps P to Q.

Notice that $C(3, \frac{3}{2})$ is the midpoint of the line segment $\overline{PQ}$ and $y = -2x + \frac{15}{2}$ is the equation for the line through C and perpendicular to to $\overline{PQ}$ [just use standard algebra to find these].

Then we wish to find the matrix for R_ℓ, where ℓ is given by $\begin{bmatrix} 2 & 1 & -\frac{15}{2} \end{bmatrix}$.

Notice that this line intersects the x-axis at the point $C(\frac{15}{4}, 0, 1)$.

We then compute the angle between the lines $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 & -\frac{15}{2} \end{bmatrix}$:

$m(\angle h, \ell) = \tan^{-1}\left(\frac{p_1q_2 - p_2q_1}{p_1q_1 + p_2q_2}\right) = \tan^{-1}\left(\frac{0+2}{0+1}\right) = \tan^{-1}(2) \approx 63.435°$, so we take $\theta = 180° - 63.435° = 116.565°$.

Then the rotation $R_{C,\theta}$ with $C(\frac{15}{4}, 0, 1)$ and $\theta = 116.565°$ is an isometry that maps h to the line ℓ.

Recall that $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the matrix for the reflection across the line $h \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$.

Hence, the matrix of a reflection that takes P to Q is given by: $R_\ell = T \circ R_h \circ T^{-1}$, so the matrix B of R_ℓ is given by:

$B = TAT^{-1}$ where A is as above, and T is the direct isometry that maps h to ℓ. Notice that T is given by:

$R_{C,\theta} = T_{OC} R_{O,\theta} T_{CO}$ with $C(\frac{15}{4}, 0, 1)$ and $\theta = 116.565°$.

The specific computations from here are left to the reader.

(c) Find the matrix of a *rotation* that maps P to Q.

One way to obtain a rotation that takes P to Q is to consider a rotation about the point $C(3, \frac{3}{2}, 1)$ and to set $\theta = 180°$.

In this case, $T_{OC} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}$, $T_{OC} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}$, and $R_{O,\theta} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Hence $R_{C,\theta} = T_{OC} R_{O,\theta} T_{CO} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 0 & 3 \\ 0 & -1 & \frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 6 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

9. Consider the following transformation matrices:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & -5 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad E = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Which of these are the matrix of an affine transformation of $\mathbb{E}$?

From the form of the matrices, we see that all of these matrices represent affine transformations with the exception of A.

(b) Which of these are the matrix of an isometry of $\mathbb{E}$?

Recall that a transformation is an isometry if and only if its matrix representation has one of the following forms:

$$\begin{bmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} \cos\theta & \sin\theta & a \\ \sin\theta & -\cos\theta & b \\ 0 & 0 & 1 \end{bmatrix}$$

From this, we see that C, D, E and F represent isometries while A and B do not.

(c) Which of these are the matrix of an direct isometry of $\mathbb{E}$?

Direct isometries are transformations whose matrix representations have the form: $\begin{bmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{bmatrix}$

Therefore, we see that E and F represent direct isometries.

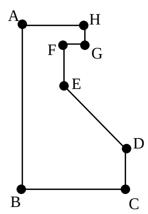
(d) Which of these are the matrix of a rotation of $\mathbb{E}$?

Rotations are direct isometries. We will see below that F is a translation, while, looking at the form of E, we see that this matrix represents a rotation (in fact, it is a rotation about O).

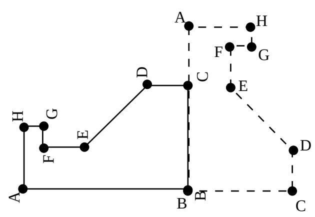
(e) Which of these are the matrix of a translation of $\mathbb{E}$?

Translations are also direct isometries. We saw above that E is a rotation. Looking at the form of F, we see that this matrix represents a translation.

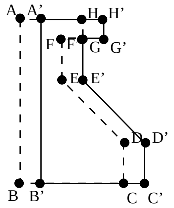
10. Given the plane figure in $\mathbb{E}$ shown below, accurately draw the image of this figure under each of the following isometries:



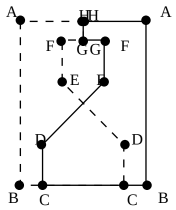
(a) $R_{B,90}$



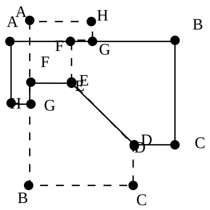
(b) T_{FG}



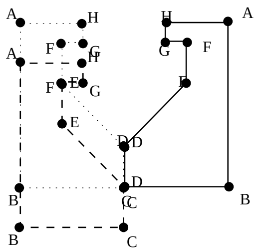
(c) R_ℓ, where $\ell = \overleftrightarrow{HG}$.



(d) R_ℓ, where $\ell = \overleftrightarrow{ED}$.



(e) G_{CD}



11. Prove or Disprove:

(a) The set of all *translations* of $\mathbb{E}$ forms a group under composition.

This statement is true. To see this, notice the following:

- The trivial transformation (the identity map) is a translation along the zero vector.
- The composition of two translations is a translation. This is due to the fact that $T_{ST} \circ T_{PQ} = T_{P'Q'}$ where $\overleftrightarrow{P'Q'} = \overleftrightarrow{PQ} + \overleftrightarrow{ST}$ (use vector addition).
- The inverse of a translation is a translation. In fact, $T_{PQ}^{-1} = T_{QP}$.
- Finally, associativity is inherited from the associativity of function composition for transformations in general.

(b) The set of all *rotations* of $\mathbb{E}$ forms a group under composition.

This statement is False. Is is true that the set of rotations about a single point in the plane form a group under composition. It is also true that the trivial map corresponds to a 0 degree rotation about a center point and that the inverse of a rotation is a rotation. However, if we compose two non-trivial rotations with different fixed points, the result is not a rotation. Therefore, the set of all rotations is not closed under composition so it cannot form a group under composition.

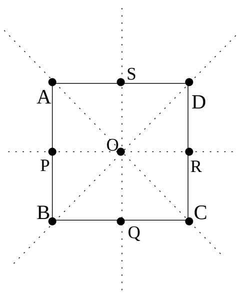
(c) The set of all *indirect isometries* of $\mathbb{E}$ forms a group under composition.

This statement is also False. While it is true that the set of all direct isometries forms a group under composition, recall that by Proposition 3.8, the composition of two affine indirect isometries is a direct isometry, so this set is not closed under composition and hence cannot form a group under composition.

12. Let $\square ABCD$ be the unit square centered at the origin in $\mathbb{E}$.

(a) Give a complete list of all of the symmetries of $\square ABCD$.

Consider the following diagram:



The identity map is clearly a symmetry of $\square ABCD$.

The rotational symmetries of $\square ABCD$ are $R_{O,90^\circ} = R$, $R_{O,180^\circ} = R^2$, $R_{O,270^\circ} = R^3$ (note that $R^4 = id$)

Let $\ell_1 = \overleftrightarrow{SQ}$, $\ell_2 = \overleftrightarrow{PR}$, $\ell_3 = \overleftrightarrow{AC}$, and $\ell_4 = \overleftrightarrow{BD}$.

Then the reflectional symmetries of $\square ABCD$ are $R_{\ell_1} = V$, $R_{\ell_2} = H$, $R_{\ell_3} = D_1$, and $R_{\ell_4} = D_2$.

These are the only symmetries of $\square ABCD$.

(b) Show that the set of isometries that you found in part (a) forms a group under composition.

We will demonstrate that this set forms a group by constructing the Cayley (multiplication table) for this set under function composition. Note that we are thinking of doing the transformation listed along the top row first followed by the transformation in the first column.

	id	R	R^2	R^3	V	H	D_1	D_2
id	id	R	R^2	R^3	V	H	D_1	D_2
R	R	R^2	R^3	id	D_1	D_2	H	V
R^2	R^2	R^3	id	R	H	V	D_2	D_1
R^3	R^3	id	R	R^2	D_2	D_1	V	H
V	V	D_2	H	D_1	id	R^2	R^3	R
H	H	D_1	V	D_2	R^2	id	R	R^3
D_1	D_1	V	D_2	H	R	R^3	id	R^2
D_2	D_2	H	D_1	V	R^3	R	R^2	id

The multiplication shown in the table above demonstrates that these transformations form a group under composition.