Math 487 – Finite Projective Geometry Handout

Axioms for a Finite Projective Plane

Undefined Terms. *point, line,* and *incident*
Axiom P1. For any two distinct points, there is exactly one line incident with both points.
Axiom P2. For any two distinct lines, there is at least one point incident with both lines.
Axiom P3. Every line has at least three points incident with it.
Axiom P4. There exist at least four distinct points of which no three are collinear.

The axiom system does not specify the number of points on a line.

Definition. A *projective plane of order n* is a geometry that satisfies the above axioms for a finite projective plane and has at least one line with exactly $n + 1$ ($n > 1$) distinct points incident with it.

Do projective planes exist? Can we produce a model?

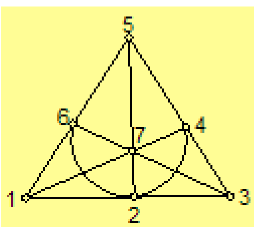

Diagram model on the left for a projective plane of order 2, points are defined by the seven dots and lines by the six straight segments and one curved segment. Note each line contains exactly three points.
The second model is illustrated in the table below.

points	lines
A, B, C, D, *E, F, G*	*ADB, AGE, AFC, BEC,* *BGF, CGD, FDE*

Next, we give a model of a projective plane of order 3. In this case, there exists at least one line with exactly 4 distinct points incident with it.

points	lines
A, B, C, D, *E, F, G, H,* *I, J, K, L, M*	*ABCD, AEFG, AHIJ, AKLM, BEHK,* *BFIL, BGJM, CEIM, CFJK, CGHL, DEJL,* *DFHM, DFIK*

Theorem P1. There exists a projective plane of order n for some positive interger n.

Question. For what values of *n*, does a projective plane exist?

The question is an unsolved problem in mathematics, though partial results have been obtained. In 1906, in a paper published in the Transactions of the American Mathematical Society, Oswald Veblen and W. Bussey proved that there exist finite projective planes of order p^m where *p* is a prime number and *m* is a positive integer. Hence, there are projective planes of orders 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, etc.

The conjecture is that the only orders for which there is a projective plane of order *n* is when *n* is a prime number to some positive integer power. A result by Bruck and Ryser in 1949 partially proved the conjecture. Their result stated that there is no projective plane of order *n*, if *n* is congruent to 1(mod 4) or 2(mod 4), and *n* cannot be written as the sum of two squares. This result showed that *n* cannot be 6, 14, 21, 22, etc. Currently, the lowest order for which the conjecture has not been proven is 12.