**Math 323**
Cylindrical and Spherical Coordinates

Cylindrical Coordinates:

Definition: Cylindrical coordinates are an alternate way of describing points in 3-space. Basically, one of the rectangular coordinate planes is replaced by a polar plane (usually the xy-plane, and we will assume this in our descriptions and formulas, but any coordinate plane would do). A point P is then given in terms of the coordinates $P(r, \theta, z)$, where θ is the angle from the positive half of the x-axis, r is the distance from the origin to the projection of P in the xy-plane, and z is the distance from P to the xy-plane.
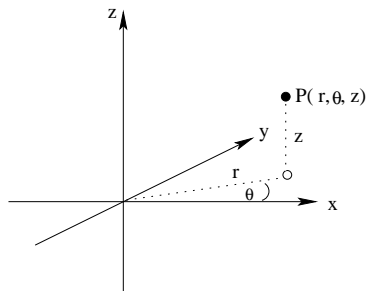
Definition 17.29: The relationship between a point $P(x, y, z)$ given in polar coordinates and the same point $P(r, \theta, z)$ given in polar coordinates is as follows:

$x = r \cos \theta$

$y = r \sin \theta$

$z = z$

$r^2 = x^2 + y^2$

$\tan \theta = \dfrac{y}{x}$.

To evaluate triple integrals, in cylindrical coordinates, we incorporate the differential for double integrals in polar coordinates $dA = r \, dr \, d\theta$ in order to obtain the differential $dV = dz \, dy \, dx = r \, dz \, dr \, d\theta$

Theorem 17.30 (Fubini for Cylindrical Coordinates):

(I) Let Q be a solid bounded "above" and "below" by a pair of surfaces $k_1(r, \theta)$ and $k_2(r, \theta)$ each sitting "over" a region R in the polar plane and suppose that $f(r, \theta, z)$ is continuous on Q, then

$$\iiint\limits_{Q} f(r, \theta, z) \, dV = \iint\limits_{R} \int_{k_1(r,\theta)}^{k_2(r,\theta)} f(r, \theta, z) \, dz \, dA$$

(II) Further suppose that the region R is bounded by polar functions $r = g_1(\theta)$ and $r = g_2(\theta)$, for $\alpha \leq \theta \leq \beta$. Then:

$$\iiint\limits_{Q} f(r, \theta, z) \, dV = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} \int_{k_1(r,\theta)}^{k_2(r,\theta)} f(r, \theta, z) \, r \, dz \, dr \, d\theta$$

Example: Set up a triple integral in cylindrcial coordinates that gives the volume of the region above $z = \sqrt{x^2 + y^2}$ and below $z = \sqrt{8 - x^2 - y^2}$

First notice that the intersection in these two surfaces is: $\sqrt{x^2 + y^2} = \sqrt{8 - x^2 - y^2}$ or $x^2 + y^2 = 8 - x^2 - y^2$, so $2x^2 + 2y^2 = 8$. Then $x^2 + y^2 = 4$, so this volume can be thought of as sitting over the circle of radius 2 about the origin in the xy-plane.

Translating everything into cylindrical coordinates, we get $k_1(r, \theta) = \sqrt{r^2} = r$ and $k_2(r, \theta) = \sqrt{8 - r^2}$ describe the two bounding surfaces, and R is given by $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$.

Thus $V = \displaystyle\int_{0}^{2\pi} \int_{0}^{2} \int_{r}^{\sqrt{8-r^2}} 1 \cdot r, dz \, dr \, d\theta$

Spherical Coordinates:

Definition: Spherical coordinates are another way of describing points in 3-space. In this system, a point $P(\rho, \phi, \theta)$ is then given in terms of two angles and a distance. The angle θ is the angle from the positive half of the x-axis to the ray from the origin to the projection of P in the xy-plane. The angle ϕ is the angle of declination from the positive z-axis to the point P, and ρ is the distance between P and the origin.
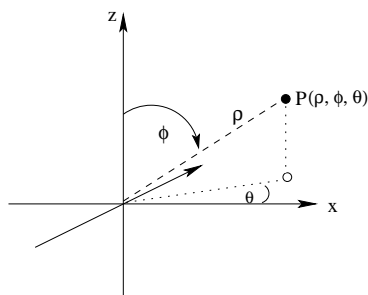
Definition 17.29: The relationship between a point $P(x, y, z)$ given in polar coordinates and the same point $P(\rho, \phi, \theta)$ given in spherical coordinates is as follows:

$x = \rho \sin \phi \cos \theta$

$y = \rho \sin \phi \sin \theta$,

$z = \rho \cos \phi$

$\rho^2 = x^2 + y^2 + z^2$.

Notice that $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$, and $\rho \geq 0$.

To evaluate triple integrals in spherical coordinates, we use the differential for spherical coordinates: $dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$.

Theorem 17.32 (Fubini for Spherical Coordinates):

(I) Let Q be a solid bounded "inside" and "outside" by the surfaces $\rho = k_1(\phi, \theta)$ and $\rho k_2(\phi, \theta)$ for $\gamma_1 \leq \phi \leq \gamma_2$) and $\alpha \leq \theta \leq \beta$). Then:

$$\iiint\limits_{Q} f(\rho, \phi, \theta) \, dV = \int_{\alpha}^{\beta} \int_{\gamma_1}^{\gamma_2} \int_{k_1(\phi,\theta)}^{k_2(\phi,\theta)} f(\rho, \phi, \theta) \, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

Example: Set up a triple integral in spherical coordinates that gives the volume of the solid bounded below by $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 2$.

First, notice that the region Q is a truncated cone. We must translate the solid region into spherical coordinates. We have $z = 2 = \rho \cos \phi$, so $\rho = \dfrac{2}{\cos \phi}$, or $\rho = 2 \sec \phi$. From this, we see that $0 \leq \rho \leq 2 \sec \phi$ on this region.

Next, since this surface sits over a circle in the xy-plane, we must have $0 \leq \theta \leq 2\pi$.

Finally, since the cone $z^2 = x^2 + y^2$ has sides bounded by the lines $z = \pm x$ and $z = \pm y$, the sides of the cone sit at $45°$ angles with both the xy-plane and the positive z-axis. Therefore, $0 \leq \phi \leq \dfrac{\pi}{4}$.

Thus the volume is given by: $V = \displaystyle\int_{0}^{2\pi} \int_{0}^{\frac{\pi}{4}} \int_{0}^{2 \sec \phi} 1 \, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$