**Math 323**
**Double Integrals**

**Recall:** In Calculus I, when we defined the definite integral, we used the idea of *partitions*, rectangular approximations, and *Riemann sums*. That is, in order to find the area under a continuous function on an interval $[a, b]$, we picked a partition of points $\mathcal{P}$ consisting of points $a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$ and then computed the Riemann sum $A \approx \sum_{i=1}^{n} f(c_i)\Delta x_i$ where $c_i \in [x_{i-1}, x_i]$ and $\Delta x_i = x_i - x_{i-1}$.

Then, we defined $\int_a^b f(x)\, dx = \lim_{\|\mathcal{P}\| \to 0} \sum f(c_i)\Delta x_i$, where $\|\mathcal{P}\|$ is the maximum value of $\Delta x_i$ for a partition and where the limit is taken over all possible refinements of all partitions of $[a, b]$ (provided this limit exists).

**Goal:** Our new goal is to extend the concept of the definite integral to functions of several variables. We will begin by discussing the idea of a definite integral of a function $f(x, y)$ over some *closed* region $R$ in the plane. We usually interpret this integral as the *volume* "under" this function of two variables and "inside" the region $R$. The development of this definite integral will be quite similar to what we did before. We will begin by approximating the volume. Here, we will think of taking the region $R$ and choosing a partition $\mathcal{P}$ of points $(x_i, y_i)$ which subdivide $R$ into rectangular sub-regions in the $xy$-plane. We then approximate the volume under the function over the region $R$ by computing the sum of the volume of the rectangular solids whose base is one of the rectangles in the partition of $R$ and whose height is the height of the function $f(x, y)$ at some point within the rectangle.

With this in mind, for any given partition $\mathcal{P}$ of $R$, we define $\|\mathcal{P}\|$ to be the maximum diagonal length among all the rectangles in $R$ formed by the partition, and the Riemann sum for such a partition $\mathcal{P}$ is $V \approx \sum_{R_{i,j}} f(u_i, v_j)\Delta A_{i,j}$, where $\Delta A_{i,j}$ is the area of a rectangle $R_{i,j}$ formed by the partition, and $(u_i, v_j)$ is some choice of point within the rectangle $R_{i,j}$.

**Definition:** Let $f$ be a function of two variables that is defined on a region $R$ in the plane. The **double integral of f over R**, denoted by $\iint_R f(x, y)\, dA = \lim_{\|\mathcal{P}\| \to 0} \sum_{R_{i,j}} f(u_i, v_j)\Delta A_{i,j}$, provided the limit exists. When this limit exists, we say that the function $f(x, y)$ is **integrable** over the region $R$.

**Definition 17.4:** Let $f$ be a continuous function of two variables such that $f(x, y) \geq 0$ for every $(x, y)$ in a region $R$. The **volume** $V$ of the solid that lies under the graph of $z = f(x, y)$ and over $R$ is: $V = \iint_R f(x, y)\, dA$

Note that we are claiming that the limit over all possible refinements of any partition of $R$ exists when $f$ is continuous. Also, as before, when $f$ takes on negative values, we need to think about the "meaning" of the function $f$ and decide whether we want to consider "negative volume" or "unsigned volume".

Now that we have this definition, it is fairly clear how it can be used to *estimate* the volume under a given function over a region $R$. What remains is to find a way to find such volumes *exactly* – that is, to find a result analogous to the FTC, only for double integrals.

Before we do this, we pause to consider some properties of double integrals which are analogous the properties of definite integrals in one variable:

**Theorem 17.5:** (i) $\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA$ for every real number $c$.

(ii) $\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$

(iii) If $R$ is the disjoint union of two sets $R_1$ and $R_2$ ($R_1 \cup R_2 = R$), then:
$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$

(iv) If $f(x, y) \geq 0$ throughout $R$, then $\iint_R f(x, y)\, dA \geq 0$

**Idea:** In order to compute the value of double integrals *exactly*, we need a new concept: an **iterated integral**. As you might have guessed, just as differentiating functions of more than one variable involved differentiating with respect to one variable at a time by defining **partial derivatives**, antidifferentiating functions of several variables will make use of **partial integration**.

To evaluate the *iterated integral* $\int_a^b \left[\int_c^d f(x,y)\ dy\right] dx$, we first antidifferentiate $f$ with respect to $y$, then we evaluate $y$ at each endpoint and simplify, and then we antidifferentiate the result with respect to $x$ and evaluate the result using the second pair of endpoints, yielding a numerical result.

**Example:** Let $f(x,y) = 2x^2y - 4xy^2$ and $R = \{(x,y)\,|\,0 \le x \le 4, 2 \le y \le 5\}$

$$\int_0^4 \left[\int_2^5 2x^2y - 4xy^2\ dy\right] dx = \int_0^4 \left( x^2y^2 - \frac{4}{3}xy^3 \Big|_{y=2}^{y=5} \right) dx = \int_0^4 \left[ \left(25x^2 - \frac{4}{3}(125)x\right) - \left(4x^2 - \frac{4}{3}(8)x\right)\right] dx$$

$$= \int_0^4 \left(21x^2 - 156x\ dx\right) = 7x^3 - 78x^2 \Big|_0^4 = (7(4)^3 - 78(4)^2) - (0-0) = -800.$$

An interesting question is what happens when we compute the iterated integral in the other order:

$$\int_2^5 \left[\int_0^4 2x^2y - 4xy^2\ dx\right] dy = \int_2^5 \left( \frac{2}{3}x^3y - 2x^2y^2 \Big|_{x=0}^{x=4} \right) dy$$

$$= \int_2^5 \left[\left(\frac{2}{3}(64)y - 2(16)y^2\right) - (0-0)\right] dy = \int_2^5 \left(\frac{128}{3}y - 32y^2\ dy\right)$$

$$= \frac{64}{3}y^2 - \frac{32}{3}y^3 \Big|_2^5 = \left(\frac{64}{3}(25) - \frac{32}{3}(125)\right) - \left(\frac{64}{3}(4) - \frac{32}{3}(8)\right) = -(800) - (0) = -800.$$

It turns out this this is *not* a coincidence. Over an rectangular region, if $f(x,y)$ is integrable over $R$, then both iterated integrals give the same value.

**Definition 17.6:**

(I) $\displaystyle\int_a^b \int_c^d f(x,y)\ dy\,dx = \int_a^b \left[\int_c^d f(x,y)\ dy\right] dx$ (II) $\displaystyle\int_c^d \int_a^b f(x,y)\ dx\,dy = \int_c^d \left[\int_a^b f(x,y)\ dx\right] dy$

**Notes:**

• This notation describes iterated integrals over a rectangular region of the plane. We work the **innermost** integral first, then the outer integral in each case.

• We can also compute iterated integrals over regions bounded either vertically by curves $g_1(x)$ and $g_2(x)$ or horizontally by curves $h_1(y)$ and $h_2(y)$:

**Definition 17.6:**

(I) $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\right] dx$ (II) $\displaystyle\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\right] dy$

Intuitively, when we compute an iterated integral, we think of slicing the volume under the curve either horizontally (or vertically) into infinitely this cross sections. The inner integral in our iterated integral can be thought of as finding a formula for the area of each cross section, and the outer integral "adds up" the area of each slice as we run through all values of that variable over the region under consideration.

**Theorem 17.8 (Fubini):**

(I) Let $R$ be a region in the plane bounded horizontally by functions $g_1(x)$ and $g_2(x)$. and suppose that $f(x,y)$ is continuous on $R$, then

$$\iint_R f(x,y)\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\,dx$$

(II) Let $R$ be a region in the plane bounded vertically by functions $h_1(y)$ and $h_2(y)$. and suppose that $f(x,y)$ is continuous on $R$, then

$$\iint_R f(x,y)\ dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\,dy$$

**Examples:**

1. Evaluate $\iint\limits_R f(x,y)\, dA$ where $f(x,y) = 2xy$, and $R$ is the region bounded by $g_1(x) = x^2 - 1$ and $g_2(x) = 1 - x^2$

2. Evaluate $\iint\limits_R f(x,y)\, dA$ where $f(x,y) = 3x^2 - 2xy$, and $R$ is the region bounded by $h_1(y) = y$ and $h_2(y) = y^2$

3. Evaluate $\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$