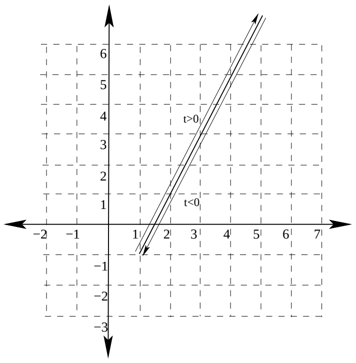
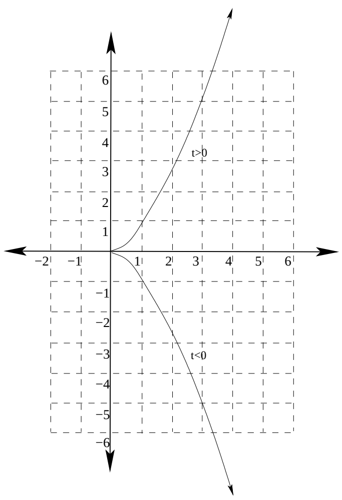
Math 323
Exam 1
Practice Problem Solutions

1. For each of the following curves, first find an equation in x and y whose graph contains the points on the curve. Then sketch the graph of C, indicating its orientation.

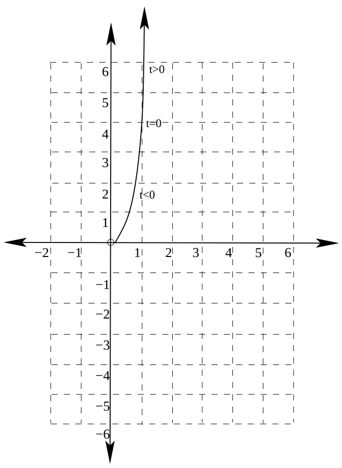
   (a) $x = t^2 + 1$ and $y = 2t^2 - 1$
   Equation: $y = 2x - 3$ [Notice $x - 1 = t^2$]



   (b) $x = t^2$ and $y = t^3$
   Equation: $y = \pm x^{\frac{3}{2}}$ [Notice $t = \pm\sqrt{x} = \pm x^{\frac{1}{2}}$]

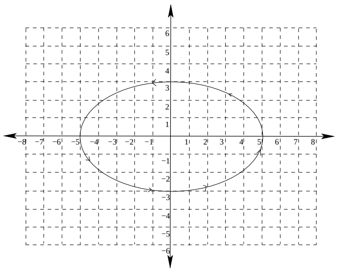


   (c) $x = e^t$, $y = 4e^{2t}$
   Equation: $y = 4x^2$ [Notice $y = 4\left(e^t\right)^2$]



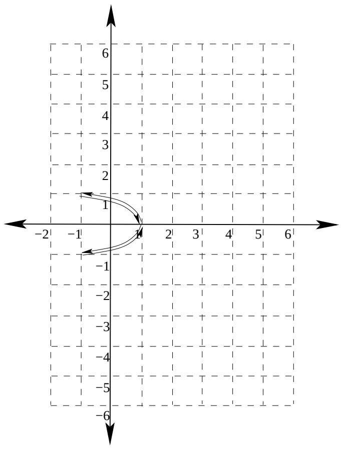
(d) $x = 5\cos t$, $y = 3\sin t$

Equation: $\frac{x^2}{25} + \frac{y^2}{9} = 1$ [Use: $\sin^2 t + \cos^2 t = 1$]



(e) $x = \cos 2t$, $y = \sin t$

Equation: $x = 1 - 2y^2$ [Use: $\cos(2t) = 1 - 2\sin^2 t$]



2. Given the parametric curve: $x = 2t^2 + 1$, $y = 3t^3 + 2$

(a) Find the equation of the tangent line to this parametric curve corresponding when $t = 1$.

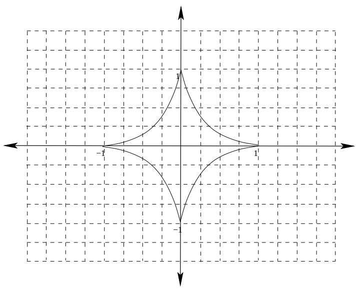
Recall that $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{9t^2}{4t} = \frac{9}{4}t$, which when $t = 1$ gives $m = \frac{9}{4}$.

Also, when $t = 1$, the point on the curve is $(3, 5)$. Therefore the equation for the tangent line to the graph of this parameterized curve is given by: $y - 5 = \frac{9}{4}(x - 3)$.

(b) Compute $\frac{d^2y}{dx^2}$ and use this to determine the concavity of the curve when $t = 1$.

$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{9}{4}}{4t} = \frac{9}{16t}$, which when $t = 1$ gives $\frac{9}{16}$, so the graph of the parametric curve is concave up when $t = 1$.

3. (a) Graph the parametric curve given by $x = \cos^3 t$, $y = \sin^3 t$.



(b) Prove that the graph of this parametric curve satisfies the equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.

Notice that $x^{\frac{2}{3}} = \cos^2 t$ and $y^{\frac{2}{3}} = \sin^2 t$. Thus $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.

4. Find the arc length of the curve given by:

   (a) $x = 2t$, $y = \frac{2}{3}t^{\frac{3}{2}}$, $5 \le t \le 12$

   Notice that $x'(t) = 2$ and $y'(t) = t^{\frac{1}{2}}$.

   Therefore, $L = \displaystyle\int_5^{12} \sqrt{4+t}\, dt = \frac{2}{3}(4+t)^{\frac{3}{2}}\Big|_5^{12} = \frac{2}{3}\left[(16)^{\frac{3}{2}} - (9)^{\frac{3}{2}}\right] = \frac{2}{3}(64-27) = \frac{74}{3}$

   (b) $x = \sin t - \cos t$, $y = \sin t + \cos t$, $\frac{\pi}{4} \le t \le \frac{\pi}{2}$

   Notice that $x'(t) = \cos t + \sin t$ and $y'(t) = \cos t - \sin t$

Then $L = \displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{(\cos t + \sin t)^2 + (\cos t - \sin t)^2}\, dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{\cos^2 t + 2\sin t\cos t - \sin^2 t + \cos^2 t - 2\sin t \cos t + \sin^2 t}\, dt$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{2}\, dt = \sqrt{2}t\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \sqrt{2}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{\sqrt{2}\pi}{4}$$

5. Find the surface area of the solid obtained by revolving the parameterized curve given by revolving $x = t^3$, $y = 2t+3$ for $0 \le t \le 1$ about the y-axis.

   $S = \displaystyle\int_0^1 2\pi x\, ds = \int_0^1 2\pi t^3\sqrt{(3t^2)^2 + (2)^2}\, dt = 2\pi \int_0^1 t^3\sqrt{9t^4+4}\, dt$

   If we let $u = 9t^4 + 4$, then $du = 36t^3\, dx$, or $\frac{1}{36}du = t^3\, dx$.

   Then we have $S = 2\pi \displaystyle\int_4^{13} \frac{1}{36}\sqrt{u}\, du = \frac{\pi}{27}\left(\frac{2}{3}\right)u^{\frac{3}{2}}\Big|_4^{13} = \frac{\pi}{27}\left[13^{\frac{3}{2}} - 8\right]$

6. Express the following polar equations in rectangular coordinates:

   (a) $r = -5\cos\theta$

   Multiplying both sides by r gives $r^2 = -5r\cos\theta$.

   Then $x^2 + y^2 = -5x$, or $x^2 + 5x + y^2 = 0$

   Completing the square, $(x + \frac{5}{2})^2 + y^2 = \frac{5}{4}$

   (b) $r = \sin(2\theta)$

   By the double angle identity for $\sin\theta$, $r = 2\sin\theta\cos\theta$.

   Then $r^3 = 2r\sin\theta r\cos\theta$, or $(x^2+y^2)^{\frac{3}{2}} = 2xy$

   Then $(x^2+y^2)^3 = 4x^2y^2$

7. Express the following rectangular equations in polar coordinates:

   (a) $xy = 1$

   $r\cos\theta r\sin\theta = 1$, or $r^2 = \frac{1}{\sin\theta\cos\theta}$

   Then $r^2 = \sec\theta\csc\theta$

   (b) $x^2 - y^2 = 1$

   $r^2\cos^2\theta - r^2\sin^2\theta = 1$, or $r^2(\cos^2\theta - \sin^2\theta) = 1$

   By the double angle identity for $\cos\theta$, $r^2\cos(2\theta) = 1$.

   Then $r^2 = \frac{1}{\cos(2\theta)}$, or $r^2 = \sec(2\theta)$.

8. Find the equation for a circle with center $(0, -4)$ and passing through the origin in both rectangular and polar coordinates.

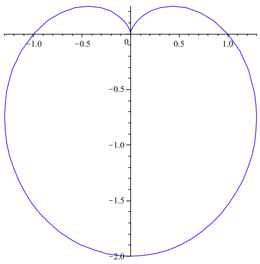
   Rectangular: $x^2 + (y+4)^2 = 16$

   Expanding, $x^2 + y^2 + 8y + 16 = 16$, or $x^2 + y^2 = -8y$

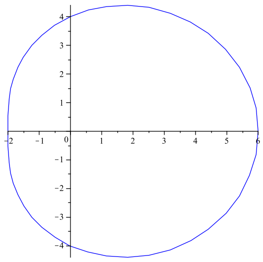
   Then $r^2 = -8r\sin\theta$, or $r = -8\sin\theta$

9. Graph each of the following polar equations:

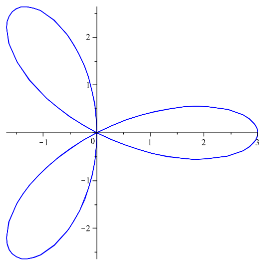
(a) $r = 1 - \sin\theta$



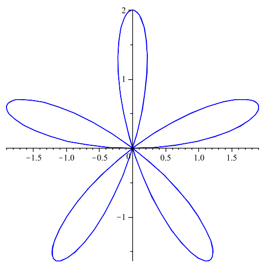
(b) $r = 4 + 2\cos\theta$



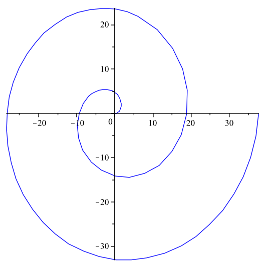
(c) $r = 3\cos(3\theta)$



(d) $r = 2\sin(5\theta)$

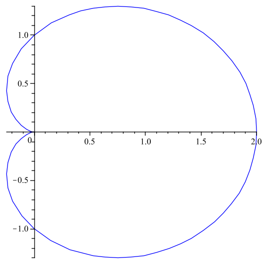


(e) $r = 3\theta$



10. Find the area of each of the following polar regions:

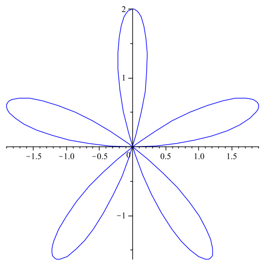
(a) the region bounded by the polar graph $r = 1 + \cos\theta$



$$A = 2\int_0^{\pi} \frac{1}{2}(1+\cos\theta)^2\ d\theta = \int_0^{\pi} 1 + 2\cos\theta + \cos^2\theta\ d\theta$$

$$= \int_0^{\pi} 1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos(2\theta)\ d\theta = \frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin(2\theta)\Big|_0^{\pi} = \frac{3\pi}{2}$$

(b) the region bounded by one loop of the curve $r = 2\sin(5\theta)$

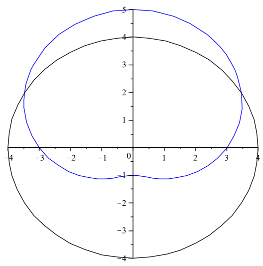


Notice that $2\sin(5\theta) = 0$ when $5\theta = \pi k$ or when $\theta = \frac{\pi k}{5}$. Then the first loop is traced on when $0 \le \theta \le \frac{\pi}{5}$.

Therefore, $A = \displaystyle\int_0^{\frac{\pi}{5}} \frac{1}{2}(2\sin(5\theta))^2\ d\theta = \int_0^{\frac{\pi}{5}} 2\sin^2(5\theta)\ d\theta$

$$= 2\int_0^{\frac{\pi}{5}} \frac{1}{2} - \frac{1}{2}\cos(10\theta)\ d\theta = 2\left[\frac{1}{2}\theta - \frac{1}{20}\sin(10\theta)\right]\Bigg|_0^{\frac{\pi}{5}} = \frac{\pi}{5}$$

(c) the region inside $r = 3 + 2\sin\theta$ and outside $r = 4$



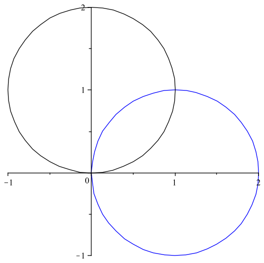
First notice that $3 + 2\sin\theta = 4$ when $2\sin\theta = 1$, or $\sin\theta = \frac{1}{2}$. That is, when $\theta = \frac{\pi}{6} + 2k\pi$ or $\frac{5\pi}{6} + 2k\pi$.

Then, using symmetry, $A = 2\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}((3+2\sin\theta)^2 - (4)^2)\ d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9 + 12\sin\theta + 4\sin^2 - 16\ d\theta$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} -7 + 12\sin\theta + 4\left(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right)\ d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} -5 + 12\sin\theta - 2\cos(2\theta)\ d\theta$$

$$= -5\theta - 12\cos\theta - \sin(2\theta)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \left(-\frac{5\pi}{2}\right) - \left(-\frac{5\pi}{6} - (12)\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = \frac{13\sqrt{3}}{2} - \frac{5\pi}{3}$$

(d) the region inside both $r = 2\cos\theta$ and $r = 2\sin\theta$



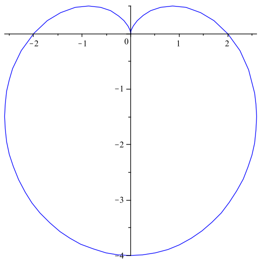
Notice that $2\cos\theta = 2\sin\theta$ when $\tan\theta = 1$, or when $\theta = \frac{\pi}{4} + k\pi$. Also notice that $2\sin\theta$ is the boundary curve for $0 \le \theta \le \frac{\pi}{4}$ while $2\cos\theta$ is the boundary curve for $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$.

Then $A = \displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{2}(2\sin\theta)^2\ d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}(2\cos\theta)^2\ d\theta = \int_0^{\frac{\pi}{4}} 2\sin^2\theta\ d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\cos^2\theta\ d\theta$

$$= \int_0^{\frac{\pi}{4}} 1 - \cos(2\theta)\ d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 + \cos(2\theta)\ d\theta = \theta - \frac{1}{2}\sin(2\theta)\Big|_0^{\frac{\pi}{4}} + \theta + \frac{1}{2}\sin(2\theta)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left[\left(\frac{\pi}{4} - \frac{1}{2}\right) - (0)\right] + \left[\left(\frac{\pi}{2} + 0\right) - \left(\frac{\pi}{4} + \frac{1}{2}\right)\right] = \frac{\pi}{2} - 1$$

11. Find the arc length of the polar curve $r = 2 - 2\sin\theta$



Notice that $r'(\theta) = -2\cos\theta$ and recall that $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

Setting up the integral is fairly routine: Using symmetry, $L = 2\displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{(2 - 2\sin\theta)^2 + (-2\cos\theta)^2}\ d\theta$

$$= 2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{4 - 8\sin\theta + 4\sin^2\theta + 4\cos^2\theta}\ d\theta$$

$= 2\displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{8 - 8\sin\theta}\ d\theta$, which, by the co-function identity for $\sin\theta$ and $\cos\theta$

$$= 2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{8 - 8\cos(\frac{\pi}{2} - \theta)}\ d\theta = 2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{16\left(\frac{1}{2} - \frac{1}{2}\cos(\frac{\pi}{2} - \theta)\right)}\ d\theta$$

Which, using the half angle identity, $= 2\displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{16\sin^2(\frac{\pi}{4} - \frac{1}{2}\theta)}\ d\theta = 8\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} |\sin(\frac{\pi}{4} - \frac{1}{2}\theta)|\ d\theta$

$$= 16|\cos(\frac{\pi}{4} - \frac{1}{2}\theta)|\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = 16|\cos(\frac{\pi}{4} - \frac{3\pi}{4}\theta) - 16\cos(\frac{\pi}{4} - \frac{\pi}{4}\theta)| = 16$$

12. Given the vectors $\vec{a} = \langle 3, -2\rangle$ and $\vec{b} = \langle 2, 1\rangle$, find:

(a) $\vec{a} + \vec{b} = \langle 5, -1\rangle$

(b) $2\vec{a} - 3\vec{b} = \langle 6, -4\rangle - \langle 6, 3\rangle = \langle 0, -7\rangle$

(c) $\|3\vec{a} - \vec{b}\| = \|\langle 9, -6\rangle - \langle 2, 1\rangle\| = \|\langle 7, -7\rangle\| = \sqrt{49 + 49} = 7\sqrt{2}$

(d) a unit vector in the direction of $\vec{a}-\vec{b}$
$\vec{a}-\vec{b}=\langle 1,-3\rangle$, and $\|\langle 1,-3\rangle\|=\sqrt{10}$.
Hence our unit vector is: $\langle \frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\rangle$

(e) a vector with magnitude 6 in the direction of $2\vec{a}+\vec{b}$
$2\vec{a}+\vec{b}=\langle 6,-4\rangle+\langle 2,1\rangle=\langle 8,-3\rangle$, and $\|\langle 8,-3\rangle\|=\sqrt{64+9}=\sqrt{73}$.
Hence the desired vector is: $6\langle \frac{8}{\sqrt{73}}, \frac{-3}{\sqrt{73}}\rangle=\langle \frac{48}{\sqrt{73}}, \frac{-18}{\sqrt{73}}\rangle$

13. Suppose that the thrust of an airplane's engine produces a speed on 400mph in still air and wind velocity is given by $\langle -20,30\rangle$. Find the direction the plane should head in order to fly due north. Also find the speed at which the plane will travel this course.

Let $\vec{v}=\langle x,y\rangle$ be the velocity vector of the airplane and $\vec{w}=\langle -20,30\rangle$ be the wind vector. Since the airplane ends up flying due north, we know that $\vec{v}+\vec{w}=\langle 0,s\rangle$. Therefore, $\vec{v}=\langle 20,s-30\rangle$. Let $y=s-30$

Now, $\|\vec{v}\|=400=\sqrt{20^2+y^2}=\sqrt{400+y^2}$, so $400^2=160,000=400+y^2$. Thus $159,600=y^2$, so $y=\sqrt{159600}=20\sqrt{399}$.

Notice that $\arctan\frac{20\sqrt{399}}{20}=87.13°$, so the plane should head $2.87°$ East of North.

Finally, the speed of the airplane is $s=20\sqrt{399}+30\approx 429.5$ miles per hour.

14. The water from a fire hose exerts a force of 200 lbs on the person holding the hose. The nozzle of the hose weighs 20 lbs. Find the force required to hold the hose horizontal and find the angle to the horizontal that this force must be applied.

Notice that the total force acting on the hose (from the water presure combined with gravity) is $\vec{h}=\langle -200,-20\rangle$.

Then the force required to keep the hose level acts equal and opposite these forces: $\vec{v}=\langle 200,20\rangle$

Hence $\|\vec{v}\|=\sqrt{200^2+20^2}=\sqrt{40400}=20\sqrt{101}$ lbs of force.

The angle above the horizontal at which this force must be applied is given by: $\theta=\arctan(\frac{20}{200})\approx 5.7°$

15. Given $P(-1,0,2)$ and $Q(1,2,-4)$

(a) Plot P and Q.
(I'll let you figure out this one).

(b) Find $d(P,Q)=\sqrt{(2)^2+(2)^2+(-6)^2}=\sqrt{4+4+36}=\sqrt{44}=2\sqrt{11}$.

(c) Find the midpoint of the line segment between P and Q.
$M=\left(\frac{-1+1}{2}, \frac{0+2}{2}, \frac{2+(-4)}{2}\right)=(0,1,-1)$

(d) Find the equation for the sphere centered at P and passing through Q.
$(x+1)^2+y^2+(z-2)^2=44$

16. Given the vectors $\vec{v}=2i-j+3k$ and $\vec{w}=4i+3j-k$

(a) Find $\vec{v}+\vec{w}=6i+2j+2k$

(b) Find $2\vec{v}-3\vec{w}=(4i-2j+6k)-(12i+9j-3k)=-8i-11j+9k$

(c) Find $\|\vec{w}-\vec{v}\|=\|\ 2i+4j-4k\|=\sqrt{4+16+16}=\sqrt{36}=6$

(d) Find a unit vector in the same direction as $\vec{v}$.
Since $\|\vec{v}\|=\|\ 2i-j+3k\|=\sqrt{4+1+9}=\sqrt{14}$, the unit vector we are looking for is: $\frac{2}{\sqrt{14}}i-\frac{1}{\sqrt{14}}j+\frac{3}{\sqrt{14}}k$

(e) Find a vector with magnitude 5 in the opposite direction as $\vec{w}$.
Since $\|\vec{w}\|=\|\ 4i+3j-k\|=\sqrt{16+9+1}=\sqrt{26}$, the vector we are looking for is: $-\frac{20}{\sqrt{26}}i-\frac{15}{\sqrt{26}}j+\frac{5}{\sqrt{26}}k$

17. Find the force needed to keep a helicopter stationary if the helicopter weighs 1000 lbs and a northeasterly wind exerts a force of 150 lbs on the helicopter.

$\vec{F_g} = \langle 0, 0, -1000 \rangle$ is the force of gravity on the helicopter (in lbs).

$\vec{F_w} = \langle 150\cos(135°), 150\sin(135°), 0 \rangle = \langle 75\sqrt{2}, 75\sqrt{2}, 0 \rangle$ is the force due to the wind (in lbs).

Then, to oppose these forces and keep the helicopter stationary, the helicopter must supply: $\vec{F_h} = \langle -75\sqrt{2}, -75\sqrt{2}, 1000 \rangle$.

That is, $75\sqrt{2}$ lbs West, $75\sqrt{2}$ lbs South, and 1000 lbs Up.

The magnitude of the force that must be generated by the helicopter is:

$\sqrt{(75\sqrt{2})^2 + (75\sqrt{2})^2 + (1000)^2} = \sqrt{11,250 + 11,250 + 1,000,000} = \sqrt{1,022,500} \approx 1011.187$