Math 323
Exam 1 **Name:**________________________________

Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

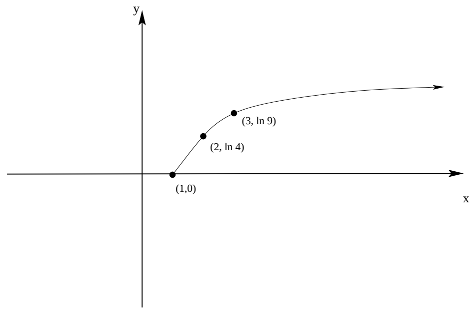
1. Given the curve: $\mathcal{C} = \begin{cases} x = \sqrt{t} \\ y = \ln t \end{cases}$ for $t \geq 1$

   (a) (6 points) Find an *explicit equation* for the underlying equation in terms of x and y containing the graph of C.

   Since $x = \sqrt{t}$, then $x^2 = t$. Substituting this gives $y = \ln x^2$, or $y = 2\ln x$, which is an explicit equation relating x and y.

   (b) (8 points) Graph $\mathcal{C}$, indicating the orientation and labeling at least three points on the graph.

t	x	y
1	1	0
2	$\sqrt{2} \approx 1.414$	$\ln 2 \approx 0.693$
4	2	$\ln 4 \approx 1.386$
9	3	$\ln 9 \approx 2.197$



   (c) (8 points) Find the equation of the *tangent line* to $\mathcal{C}$ when $t = 1$.

   Recall that $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$. Also, $\frac{dy}{dt} = \frac{1}{t}$ and $\frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}}$.

   Then $\dfrac{dy}{dx} = \dfrac{\frac{1}{t}}{\frac{1}{2}t^{-\frac{1}{2}}} = \dfrac{2t^{\frac{1}{2}}}{t} = \dfrac{2}{\sqrt{t}}$. When $t = 1$, $m = \frac{2}{\sqrt{1}} = 2$.

   Also, when $t = 1$, $x = 1$ and $y = 0$.

   Therefore, the tangent line to the curve when $t = 1$ is given by: $y - 0 = 2(x - 1)$ or $y = 2x - 2$.

   (d) (8 points) Set up (But DO NOT evaluate) an integral with respect to t representing the arc length of C for $1 \leq t \leq 4$.

   Recall that $L = \displaystyle\int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dy}\right)^2}\, dt.$

   Then $L = \displaystyle\int_1^4 \sqrt{\left(\frac{1}{2}t^{-\frac{1}{2}}\right)^2 + \left(\frac{1}{t}\right)^2}\, dt = \int_1^4 \sqrt{\frac{1}{4t} + \frac{1}{t^2}}\, dt.$

2. (6 points) Find an equation *in rectangular coordinates* for the polar equation: $r = 2\sin\theta - 4\cos\theta$

   Recall that $x^2 + y^2 = r^2$, $x = r\cos\theta$, and $y = r\sin\theta$.

   Multiplying the given equation on both sides by r gives: $r^2 = 2r\sin\theta - 4r\cos\theta$

   Then substituting gives: $x^2 + y^2 = 2y - 4x$.

   Simplifying, we get $x^2 + 4x + y^2 - 2y = 0$, or $x^2 + 4x + 4 + y^2 - 2y + 1 = 4 + 1$.

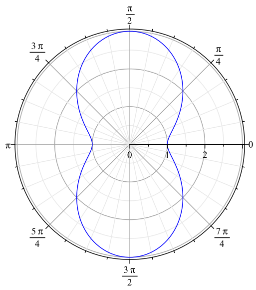
   Then $(x+2)^2 + (y-1)^2 = 5$.

3. (4 points) Describe the graph of this equation in words.

   From the form of the equation we found above, we see that this is s circle of radius $\sqrt{5}$ centered at the point $(-2, 1)$

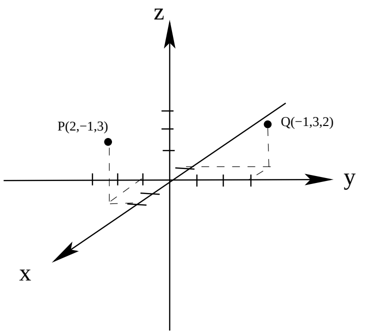
4. (12 points) **Draw the graph** of the polar equation $r = 2 - \cos(2\theta)$. Be sure to indicate the orientation and label *at least four points.*

θ	r
0	1
$\frac{\pi}{4}$	2
$\frac{\pi}{2}$	3
$\frac{3\pi}{4}$	2
π	1
$\frac{5\pi}{4}$	2
$\frac{3\pi}{2}$	3
$\frac{7\pi}{4}$	2
2π	1



5. Given the points: $P(2, -1, 3)$ and $Q(-1, 3, 2)$

(a) (6 points) Plot P and Q in 3-space.



(b) (4 points) Find $\overrightarrow{PQ}$

$\overrightarrow{PQ} = \langle -1-2, 3-(-1), 2-3\rangle = \langle -3, 4, 1\rangle$

(c) (6 points) Find a unit vector in the **opposite** direction as $\overrightarrow{PQ}$.

$\|\overrightarrow{PQ}\| = \sqrt{(-3)^2 + 4^2 + (-1)^2} = \sqrt{9+16+1} = \sqrt{26}$

Then a unit vector in the **opposite direction** as $\overrightarrow{PQ}$ is given by:

$$-\frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = -\frac{1}{\sqrt{26}} \cdot \langle -3, 4, -1\rangle = \left\langle \frac{3}{\sqrt{26}}, \frac{-4}{\sqrt{26}}, \frac{1}{\sqrt{26}} \right\rangle$$

(d) (6 points) Find an equation for the sphere containing both P and Q. [Hint: What is the center of this sphere?]

First, we notice that the center of the sphere is the midpoint between P and Q: $M = \left(\frac{2-1}{2}, \frac{-1+3}{2} \frac{3+2}{2}\right) = \left(\frac{1}{2}, 1, \frac{5}{2}\right)$

Next, notice that since $d(P, Q) = \sqrt{26}$, then $r = \frac{\sqrt{26}}{2}$.

Then the equation for this sphere is given by $(x - \frac{1}{2})^2 + (y-1)^2 + (z - \frac{5}{2})^2 = \frac{26}{4}$.

6. (10 points) Suppose that the pilot of an airplane sets a course due North at 400 mph. Further suppose that there is a steady wind of 50mph toward the Southeast. Find the resulting speed and direction of the airplane. You should give your answer as a speed in miles per hour along with a heading measured clockwise from due North.

Using the standard polar coordinate system, we let $\vec{p} = \langle 0, 400\rangle$ be the vector for the course of the plane, $\vec{w}$ be the vector of the wind, and $\vec{p} + \vec{w} = \vec{r}$ be the resultant vector.

Since the wind is to the southeast, let $\theta = -\frac{\pi}{4}$. Then $\vec{w} = 50\langle \cos\left(-\frac{\pi}{4}\right), \sin\left(-\frac{\pi}{4}\right)\rangle = \langle 50\frac{\sqrt{2}}{2}, 50\frac{-\sqrt{2}}{2}\rangle = \langle 25\sqrt{2}, -25\sqrt{2}\rangle$

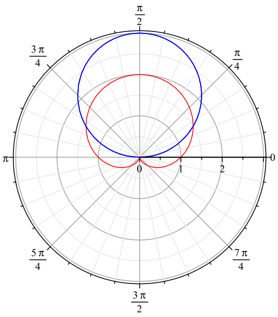
Thus $\vec{r} = \langle 0, 400\rangle + \langle 25\sqrt{2}, -25\sqrt{2}\rangle = \langle 25\sqrt{2}, 400 - 25\sqrt{2}\rangle$.

Therefore, $\|\vec{r}\| = \sqrt{(25\sqrt{2})^2 + (400 - 25\sqrt{2})^2} \approx 366.25$ miles per hour.

Moreover, $\theta = \arctan \frac{400-25\sqrt{2}}{25\sqrt{2}} \approx 84.462°$.

Therefore, the airplane should travel at the heading $90 - 84.462° = 5.538°$ East of North.

7. Consider the polar functions given by $r = 1 + \sin\theta$ and $r = 3\sin\theta$ [See the graph below].



(a) (7 points) Find the points of intersection of these two curves *exactly*.

Equating the two functions gives: $1 + \sin\theta = 3\sin\theta$, or $1 = 2\sin\theta$.
Then $\frac{1}{2} = \sin\theta$
Hence $\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$. Also notice that for these values, $r = 1 + \frac{1}{2} = \frac{3}{2}$
Therefore, the points of intersection are: $\left(\frac{\pi}{6}, \frac{3}{2}\right)$ and $\left(\frac{5\pi}{6}, \frac{3}{2}\right)$.

(b) (12 points) Set up an integral representing the area inside both of these polar curves. [You DO NOT need to evaluate your integral, and you may use any obvious symmetry to make setting up the integral more convenient]

To set up the volume integral, it is important to notice that the graphs cross at the points of intersection. On the intervals $[0, \frac{\pi}{6}]$ and $[\frac{5\pi}{6}, \pi]$, the function $r = 1 + \sin\theta$ is the bounding function, but on the interval $[\frac{\pi}{6}, \frac{5\pi}{6}]$, the function $r = 3\sin\theta$ is the bounding function for the region that we are interested in. Therefore, using symmetry, the following integral gives the area of the region inside both functions:

$$= 2\int_0^{\frac{\pi}{6}} \frac{1}{2}(3\sin\theta)^2 \, d\theta + 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}(1 + \sin\theta)^2 \, d\theta$$

Extra Credit: Sketch the graph of the polar function given by the equation $r^2 - 3r + 2 = 0$.

This equation factors to give $(r-2)(r-1) = 0$, so $r = 2$ or $r = 1$, which has the following graph: