Math 323
Exam 2 **Name:**________________________________

Instructions: You will have 60 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

1. Given the points $P(3,0,-2)$, $Q(1,2,-1)$ and $R(2,-5,3)$

   (a) (6 points) Find the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$

   First notice that $\overrightarrow{PQ} = \langle -2,2,1\rangle$ and $\overrightarrow{PR} = \langle -1,-5,5\rangle$.

   Let θ be the angle between these two vectors.

   Then $\cos\theta = \dfrac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\|\overrightarrow{PQ}\|\|\overrightarrow{PR}\|} = \dfrac{2-10+5}{\sqrt{4+4+1}\sqrt{1+25+25}} = \dfrac{-3}{3\sqrt{51}}$

   Thus $\theta = \cos^{-1}\left(\frac{-3}{3\sqrt{51}}\right) \approx 98.05°$

   (b) (6 points) Find $comp_{\overrightarrow{QP}}\overrightarrow{QR}$.

   Notice that $\overrightarrow{QP} = \langle 2,-2,-1\rangle$ and $\overrightarrow{QR} = \langle 1,-7,4\rangle$.

   Then $comp_{\overrightarrow{QP}}\overrightarrow{QR} = \dfrac{\overrightarrow{QP}\cdot\overrightarrow{QR}}{\|\overrightarrow{QP}\|} = \dfrac{2+14-4}{\sqrt{4+4+1}} = \dfrac{12}{sqrt9} = 4$

   (c) (6 points) Find a parametric equation for the line containing Q and R.

   We will use the vector $\overrightarrow{QR} = \langle 1,-7,4\rangle$ and the point $Q(1,2,-1)$ [we could have chosen to use any point on the line]. Then:

   $$\ell : \begin{cases} x = 1+t \\ y = 2-7t & t\in\mathbb{R} \\ z = -1+4t \end{cases}$$

   (d) (6 points) Find the area of $\triangle PQR$.

   Recall that the area of a triangle is given by:
   $A_\triangle = \frac{1}{2}\|\overrightarrow{PQ}\times\overrightarrow{PR}\|$

   $$\overrightarrow{PQ}\times\overrightarrow{PR} = \langle -2,2,1\rangle \times \langle -1,-5,5\rangle = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 2 & 1 \\ -1 & -5 & 5 \end{vmatrix} = \vec{\imath}(10+5) - \vec{\jmath}(-10+1) + \vec{k}(10+2) = 15\vec{\imath} + 9\vec{\jmath} + 12\vec{k}$$

   Then $A_\triangle = \frac{1}{2}\sqrt{225+81+144} = \frac{\sqrt{450}}{2} \approx 10.61$ square units.

   (e) (6 points) Find an equation for the plane containing P, Q, and R.

   First, notice that we found a normal vector to the plane in the problem above: $\overrightarrow{PQ}\times\overrightarrow{PR} = 15\vec{\imath} + 9\vec{\jmath} + 12\vec{k}$

   Then the plane, using $P : (3,0,-2)$, has equation: $15(x-3) + 9y + 12(z+2) = 0$, or $15x + 9y + 12z - 21 = 0$.

2. (10 points) Determine whether the following pair of lines are parallel, skew, or intersecting. If they intersect, find the point of intersection.

$$\ell_1 : \begin{cases} x = 3 + 2t \\ y = 3 + t \\ z = -1 + 3t \end{cases} \quad t \in \mathbb{R} \qquad\qquad \ell_2 : \begin{cases} x = 4 + s \\ y = 8 - s \\ z = -1 + 2s \end{cases} \quad s \in \mathbb{R}$$

Notice that these lines are not parallel, since their vectors, $\langle 2, 1, 3\rangle$ and $\langle 1, -1, 2\rangle$ are not scalar multiples of each other.

Next, we begin to look for a point of intersection by equating the two y coordinate functions. If $3 + t = 8 - s$, then $t = 5 - s$.

Using this, equating the x equations and substituting, $3 + 2t = 4 + s$ becomes $3 + 2(5 - s) = 4 + s$,

so $3 + 10 - 2s = 4 + s$, or, gathering terms, $9 = 3s$. Then $s = 3$. Therefore $t = 5 - (3)$ or $t = 2$

Evaluating the first line at $t = 2$ gives $(3 + 2(2), 3 + 2, -1 + 3(2)) = (7, 5, 5)$. Evaluating the second line at $s = 3$ gives $(4 + 3, 8 - 3, -1 + 2(3)) = (7, 5, 5)$.

Hence these two lines intersect at the point $(7, 5, 5)$.

3. (10 points) Find the line of intersection of the planes $x + y - z = 12$ and $x - 2y + z = -4$

These planes are not parallel, since their normal vectors $\langle 1, 1, -1\rangle$ and $\langle 1, -2, 1\rangle$ are not scalar multiples of each other.

Therefore, there must be a line common to both planes.

One way to find the line of intersection is to eliminate the z variable by adding the two planar equations:

$$\begin{array}{l} x + y - z = 12 \\ x - 2y + z = -4 \\ \hline 2x - y = 8 \end{array}$$

Solving for y in this equation gives $y = 2x - 8$.

Then, substituting this expression for y into our first equation gives:

$x + (2x - 8) - z = 12$, or $3x - 8 - z = 12$. Then, solving for z, we have $z = 3x - 20$

Finally, we set $x = t$ to obtain a parameterized equation for the line of intersection: $\ell : \begin{cases} x = t \\ y = -8 + 2t \\ z = -20 + 3t \end{cases} \quad t \in \mathbb{R}$

4. Sketch at least 3 non-trivial traces, then sketch and identify the surface given by each of the following equations:

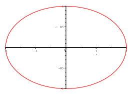
(a) (12 points) $x^2 = y^2 - 4z^2$

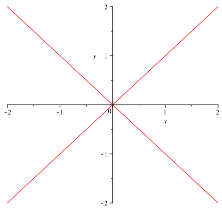
Rewriting this by moving all terms to the left hand side gives: $x^2 - y^2 - 4z^2 = 0$

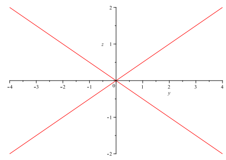
If we set $y = 0$, we get $x^2 = -4z^2$, or $x^2 + 4z^2 = 0$ which is a single point.

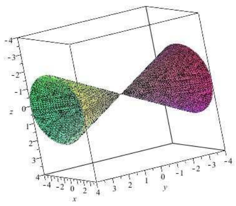
Since this is a trivial trace, we look for three other non-trivial traces.

$y = 2$ gives $4 = x^2 + 4z^2$ or $1 = \frac{x^2}{4} + z^2$ $\quad$ $z = 0$ gives $x^2 = y^2$ or $y = \pm x$ $\quad$ $x = 0$ gives $y^2 = 4z^2$ or $y = \pm 2z$









The figure is a cone with vertex at the point $(0, 0, 0)$ opening along the y-axis and with elliptical cross sections parallel to the xz-plane.

(b) (12 points) $x^2 - 1 = y^2 + 4z^2$

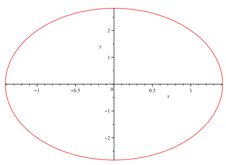
Rewriting this gives: $x^2 - y^2 - 4z^2 = 1$.

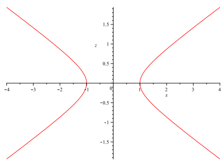
If we set $x = 0$, we get $-y^2 - 4z^2 = 1$, or $y^2 + 4z^2 = -1$ which is an empty trace.

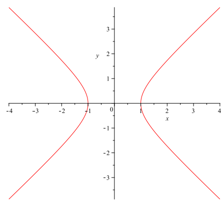
If we set $|x| = 1$, we get $1 - y^2 - 4z^2 = 1$, or $y^2 + 4z^2 = 0$, which is a single point.

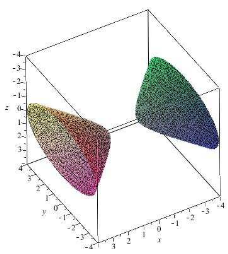
Since these are trivial traces, we look for three other non-trivial traces.

$|x| = 3$ gives $y^2 + 4z^2 = 8$ or $\frac{y^2}{8} + \frac{z^2}{2} = 1$ $\quad$ $y = 0$ gives $x^2 - 4z^2 = 1$ $\quad$ $z = 0$ gives $x^2 - y^2 = 1$







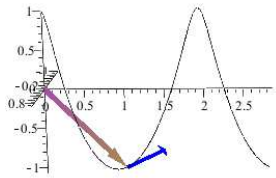


The figure is a hyperboloid of 2 sheets opening along the x-axis and with ellipical cross sections parallel to the yz-plane. beginning at the vertices $(1, 0, 0)$ and $(-1, 0, 0)$

5. Let $\vec{r}(t) = \langle \sin t, \frac{t}{\pi}, \cos t \rangle$

(a) (6 points) Sketch the graph of the curve $\mathcal{C}$ traced out by $\vec{r}(t)$ for $t \geq 0$, indicating the orientation of the curve.

t	x	y	z
0	0	0	1
$\frac{\pi}{2}$	1	$\frac{1}{2}$	0
π	0	1	-1
$\frac{3\pi}{2}$	-1	$\frac{3}{2}$	0
2π	0	2	1



(b) (6 points) Compute the vectors $\vec{r}(\pi)$ and $\vec{r}\,'(\pi)$, then add them to your the graph above.

$\vec{r}(\pi) = \langle 0, 1, -1 \rangle$
$\vec{r}\,'(t) = \langle \cos t, \frac{1}{\pi}, -\sin t \rangle$, so $\vec{r}\,'(\pi) = \langle -1, \frac{1}{\pi}, 0 \rangle$
See the diagram above. Notice that $\vec{r}(\pi)$ is a position vector with initial point at the origin, while $\vec{r}\,'(\pi)$ is a vector with initial point at the terminal point of the vector $\vec{r}(\pi)$.

(c) (6 points) Find a parametric equation for the tangent line to $\mathcal{C}$ when $t = \frac{\pi}{2}$.

We will use the vector $\vec{r}\,'(\frac{\pi}{2}) = \langle 0, \frac{1}{\pi}, -1 \rangle$ and the point $P(1, \frac{1}{2}, 0)$ [the terminal point of the vector $\vec{r}(\frac{\pi}{2}) = \langle 1, \frac{1}{2}, 0 \rangle$].
Then:

$$\ell : \begin{cases} x = 1 + 0t \\ y = \frac{1}{2} + \frac{1}{\pi}t & t \in \mathbb{R} \\ z = 0 - t \end{cases}$$

6. Suppose that an athlete throws a shot put at an angle of 30° to the horizontal at in initial speed of 40 ft/sec, and that it leaves his hand 6 feet above the ground. Recall that gravity is $32 ft/sec^2$. You may assume that gravity is the only force acting on the shot put other than its initial velocity.

(a) (7 points) Find equations for $\vec{a}(t)$, $\vec{v}(t)$, and $\vec{r}(t)$.

$\vec{a}(t) = \langle 0, -32 \rangle$.

$\vec{v}(t) = \langle 0, -32t \rangle + \langle 40 \cos 30°, 40 \sin 30° \rangle = \langle 20\sqrt{3}, 20 - 32t \rangle$.

$\vec{r}(t) = \langle 20\sqrt{3}t, 20t - 16t^2 \rangle + \langle 0, 6 \rangle = \langle 20\sqrt{3}t, 6 + 20t - 16t^2 \rangle$.

(b) (7 points) Use the equations you found in part (a) above to compute the maximum height and the horizontal range of the shot put.

The max height occurs when the $\vec{\jmath}$ component of $\vec{v}(t)$ is zero. That is, when $20 - 32t = 0$, or when $t = \frac{20}{32} = \frac{5}{8}$ seconds.

Then the max height is $6 + 20\left(\frac{5}{8}\right) - 16\left(\frac{5}{8}\right)^2 = 12.25$ feet.

The projectile lands when the $\vec{\jmath}$ component of $\vec{r}(t)$ is zero. That is, when $6 + 20t - 16t^2 = 0$, or $(4t+1)(-4t+6) = 0$
That is, when $t = -\frac{1}{4}$ or when $t = \frac{6}{4} = \frac{3}{2}$ seconds. We reject the negative solution and take $t = \frac{3}{2}$.

Then the horizontal range is found by evauating the $\vec{\imath}$ component of $\vec{r}(t)$ when $t = \frac{3}{2}$ seconds.
This gives $20\sqrt{3}(1.5) \approx 51.96$ feet.