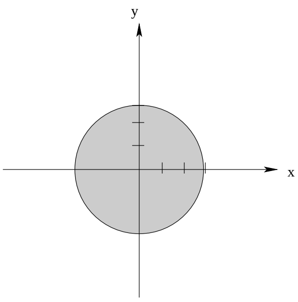
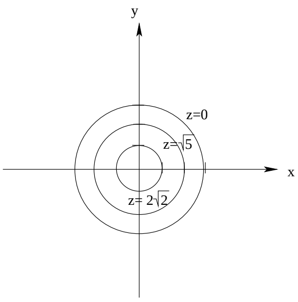
Math 323
Exam 3 - Practice Problem Solutions

1. Let $f(x,y) = \sqrt{9 - x^2 - y^2}$.

   (a) Sketch the domain of f in the x, y-plane.
   We need $9 - x^2 - y^2 \geq 0$, or $9 \geq x^2 + y^2$, so the domain is the set off all points on or inside the circle of radius 3 centered at the origin.

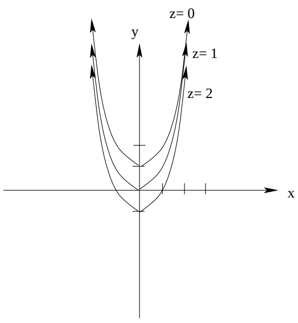


   (b) Graph contours for $z = f(x,y)$ for $z = 0, \sqrt{5}$,and $2\sqrt{2}$.
   Contours:
   If $z = 0 = \sqrt{9 - x^2 - y^2}$, then $0 = 9 - x^2 - y^2$, or $x^2 + y^2 = 9$.
   If $z = sqrt5 = \sqrt{9 - x^2 - y^2}$, then $5 = 9 - x^2 - y^2$, or $x^2 + y^2 = 4$.
   If $z = 2\sqrt{2} = \sqrt{9 - x^2 - y^2}$, then $8 = 9 - x^2 - y^2$, or $x^2 + y^2 = 1$.



2. Given the function $z = f(x,y) = 1 + x^2 - y$:

   (a) Sketch contours for this function for $z = 0, 1, 2$
   If $z = 0 = 1 + x^2 - y$, then $y = x^2 + 1$
   If $z = 1 = 1 + x^2 - y$, then $y = x^2$
   If $z = 2 = 1 + x^2 - y$, then $y = x^2 - 1$
   The contours are all parabolas.

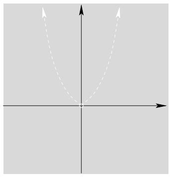


   (b) What type of curves are the x-cross sections and the y-cross sections of f?
   If $x = k$, then $z = 1 + k^2 - y$, which is a line of slope -1.
   If $y = k$, then $z = 1 + x^2 - k$, which is a parabola.

3. Sketch the domain of the following functions:

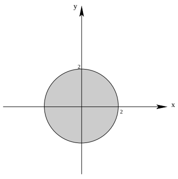
   (a) $f(x, y) = \dfrac{3xy}{y - x^2}$

   Notice that this function is defined except when $y = x^2$, therefore, the domain is the entire plane except for this parabola.

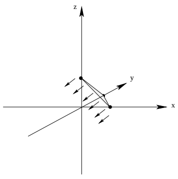
   (b) $f(x, y) = \sqrt{4 - x^2 - y^2}$

   For this function to be defined, we need $4 - x^2 - y^2 \geq 0$, or $4 \geq x^2 + y^2$. Therefore, the domain of this function is all the points either on or inside the circle or radius 2 centered at the origin.

   (c) $f(x, y, z) = \ln(1 - x - y - z)$

   For this function to be defined, we need $1 - x - y - z > 0$, or $1 > x + y + z$. Therefore, the domain of this function is all the points on the same side of the plane $x + y + z = 1$ as the origin.

4. Compute the following limits:

   (a) $\displaystyle\lim_{(x,y)\to(2,-1)} \frac{x + y}{x^2 - 2xy} = \frac{2 - 1}{2^2 - 4(2)(-1)} = \frac{1}{8}$

   (b) $\displaystyle\lim_{(x,y)\to(2,-2)} \frac{x + y}{x^2 + xy - x - y}$

   $\displaystyle = \lim_{(x,y)\to(2,-2)} \frac{x + y}{(x + y)(x - 1)} = \lim_{(x,y)\to(2,-2)} \frac{1}{x - 1} = \frac{1}{2 - 1} = 1$

   (c) $\displaystyle\lim_{(x,y,z)\to(1,1,2)} e^{\frac{x+y-z}{x+z}} = e^0 = 1$

5. Show that the following limits do not exist:

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + 2y^2}$

   First, we compute the limit as $(x, y) \to (0, 0)$ along $x = 0$:

   $\displaystyle\lim_{(0,y)\to(0,0)} \frac{2(0)y}{0^2 + 2y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{2y^2} = 0$

   Next, we compute the limit as $(x, y) \to (0, 0)$ along $x = y$:

   $\displaystyle\lim_{(x,x)\to(0,0)} \frac{2x^2}{x^2 + 2x^2} = \lim_{(0,y)\to(0,0)} \frac{2x^2}{3x^2} = \frac{2}{3}$

   Since the limits along these paths do not agree, the original limit does not exist.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y\sin x}{x^2+y^2}$

First, we compute the limit as $(x, y) \to (0, 0)$ along $x = 0$:

$$\lim_{(0,y)\to(0,0)} \frac{y\sin(0)}{0^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{y^2} = 0$$

Next, we compute the limit as $(x, y) \to (0, 0)$ along $x = y$:

$$\lim_{(x,x)\to(0,0)} \frac{x\sin x}{2x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$$

Note that the last step resulted from applying L'Hôpital's Rule the the resulting single variable limit. Since the limits along these paths do not agree, the original limit does not exist.

(c) $\displaystyle\lim_{(x,y)\to(2,0)} \frac{2y^2}{(x-2)^2+y^2}$

First, we compute the limit as $(x, y) \to (0, 0)$ along $x = 2$:

$$\lim_{(2,y)\to(2,0)} \frac{2y^2}{0^2+y^2} = \lim_{(2,y)\to(0,0)} \frac{2y^2}{y^2} = 2$$

Next, we compute the limit as $(x, y) \to (0, 0)$ along $y = 0$:

$$\lim_{(x,0)\to(0,0)} \frac{0}{(x-2)^2+0^2} = \lim_{(x,0)\to(0,0)} \frac{0}{(x-2)^2} = 0$$

Since the limits along these paths do not agree, the original limit does not exist.

(d) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^3+y^3+z^3}$

First, we compute the limit as $(x, y) \to (0, 0)$ along $x = 0$:

$$\lim_{(0,y,z)\to(0,0)} \frac{(0)yz}{0^3+y^3+z^3} = \lim_{(0,y,z)\to(0,0)} \frac{0}{y^3+z^3} = 0$$

Next, we compute the limit as $(x, y) \to (0, 0)$ along $x = y = z$:

$$\lim_{(x,x,x)\to(0,0)} \frac{x^3}{3x^3} = \frac{1}{3}$$

Since the limits along these paths do not agree, the original limit does not exist.

6. Determine all points at which the following functions are continuous:

(a) $f(x, y) = \ln(3 - x^2 + y)$

This function is continuous whenever $3 - x^2 + y > 0$. That is, when $y > x^2 - 3$, or, for all points in the plane above the parabola $y = x^2 - 3$.

(b) $f(x, y) = \tan(x + y)$

This function is continuous except when $x + y = \frac{\pi}{2} + k\pi$. That is, when $y \neq -x + \frac{\pi}{2} + k\pi$ for some integer k. Thus f is continuous except on this infinite collection of lines of slope -1.

(c) $f(x, y, z) = 4xe^{y-z}$

This function is continuous everywhere.

7. Let $f(x, y) = x^2 \sin(xy) - 3y^3$. Find f_x, f_y, f_{xy} and f_{yxy}

$f_x = 2x\sin(xy) + x^2 y\cos(xy)$.

$f_y = x^3 \cos(xy) - 9y^2$

$f_{xy} = 2x^2 \cos(xy) + x^2 \cos(xy) - x^3 y\sin(xy) = 3x^2 \cos(xy) - x^3 y\sin(xy)$

$f_{yxy} = f_{xyy} = -3x^3 \sin(xy) - x^3 \sin(xy) - x^4 y\cos(xy) = -4x^3 \sin(xy) - x^4 y\cos(xy)$

8. Let $f(x, y, x) = x^3 y^2 - \sin(yz)$. Find f_{xx} and f_{yz}

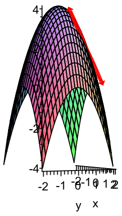
$f_x = 3x^2 y^2$, so $f_{xx} = 6xy^2$.

$f_y = 2x^3 y - z\cos(yz)$, so $f_{yz} = -\cos(yz) + yz\sin(yz)$

9. Let $f(x,y) = 4 - x^2 - y^2$. Consider the curve C formed by intersecting f with the plane $x = 1$. Find a parametric equation for the tangent line ℓ to C at the point $(1,1,2)$. Then sketch the surface given by f, the curve C and the tangent line ℓ on the same graph.

Notice that $f_y = -2y$, so when $y = 1$, the slope of the curve in the plane $x = 1$ is $m = -2$. That is, if $\Delta y = 1$, then $\Delta z = -2$ and, or course, $\Delta x = 0$ since we are in a plane parallel to the yz-plane.

Then, a parametric equation for the tangent line is given by: $\ell : \begin{cases} x = 1 \\ y = 1 + t \\ z = 2 - 2t \end{cases} \quad t \in \mathbb{R}$



10. Show that the functions $f_n(x,t) = \sin(n\pi x)\cos(n\pi ct)$ satisfy the wave equation: $c^2 \dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 f}{\partial t^2}$

Notice that $\dfrac{\partial f_n}{\partial x} = n\pi \cos(n\pi x)\cos(n\pi ct)$ and $\dfrac{\partial^2 f_n}{\partial x^2} = -n^2\pi^2 \sin(n\pi x)\cos(n\pi ct)$

On the other hand, $\dfrac{\partial f_n}{\partial t} = -n\pi c \sin(n\pi x)\sin(n\pi ct)$ and $\dfrac{\partial^2 f_n}{\partial t^2} = -n^2\pi^2 c^2 \sin(n\pi x)\cos(n\pi ct)$

Then $c^2 \dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 f}{\partial t^2}$.

11. Let $w = f(x,y) = 2x^2 - xy^2 + 3y$

(a) Find the increment Δw

$\Delta w = f(x + \Delta x, y + \Delta y) - f(x,y) = 2(x + \Delta x)^2 - (x + \Delta x)(y + \Delta y)^2 + 3(y + \Delta y) - (2x^2 - xy^2 + 3y)$
$= 2(x^2 + 2x\Delta x + (\Delta x)^2) - (x + \Delta x)(y^2 + 2y\Delta y + (\Delta y)^2) + 3y + 3\Delta y - 2x^2 + xy^2 - 3y$
$= 2x^2 + 4x\Delta x + 2(\Delta x)^2 - (xy^2 + y^2\Delta x + 2xy\Delta y + 2y\Delta y\Delta x + x(\Delta y)^2 + \Delta x(\Delta y)^2) + 3y + 3\Delta y - 2x^2 + xy^2 - 3y$
$= 4x\Delta x + 2(\Delta x)^2 - y^2\Delta x - 2xy\Delta y - 2y\Delta y\Delta x - x(\Delta y)^2 - \Delta x(\Delta y)^2 + 3\Delta y$
$= (4x - y^2)\Delta x + (3 - 2xy)\Delta y + (2\Delta x - 2y\Delta y)\Delta x + (-x\Delta y - \Delta x\Delta y)\Delta y$

(b) Find the differential dw

$dw = f_x\Delta x + f_y\Delta y = (4x - y^2)\Delta x + (-2xy + 3)\Delta y$

(c) Find $dw - \Delta w$

$dw - \Delta w = (-2\Delta x + 2y\Delta y)\Delta x + (x\Delta y + \Delta x\Delta y)\Delta y$

12. Let $w = f(x,y) = x^2\ln(y^2)$

(a) Find dw

$dw = f_x\Delta x + f_y\Delta y = (2x\ln(y^2))\Delta x + \left(x^2\frac{2y}{y^2}\right)\Delta y = (4x\ln(y))\Delta x + \left(\frac{2x^2}{y}\right)\Delta y$

(b) Use dw to approximate the change in w as the input changes from $(1,1)$ to $(1.1, 1.2)$

Notice that $\Delta x = .1$ and $\Delta y = .2$

Then $\Delta w \approx dw = (4(1)\ln(1))(.1) + \left(\frac{2(1)^2}{1}\right)(.2) = (0)(.1) + (2)(.2) = 0.4$

13. Let $w = f(x,y) = 4x^2y^3$ where $x = u^3 - v\sin u$ and $y = 4u^2 + v$. Use the Chain Rule to find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$

First notice that $f_x = 8xy^3$, $f_y = 12x^2y^2$, $x_u = 3u^2 - v\cos u$, $x_v = -\sin u$, $y_u = 8u$, and $y_v = 1$.

Then $\dfrac{\partial w}{\partial u} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial u}$

$= \left(8xy^3\right)\left(3u^2 - v\cos u\right) + \left(12x^2y^2\right)(8u) = 8(u^3 - v\sin u)(4u^2 + v)^3(3u^2 - v\cos u) + 96u(u^3 - v\sin u)^2(4u^2 + v)^2$

Similarly, $\dfrac{\partial w}{\partial v} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial v}$

$= \left(8xy^3\right)(-\sin u) + \left(12x^2y^2\right)(1) = -8(u^3 - v\sin u)(4u^2+v)^3(\sin u) + 12(u^3 - v\sin u)^2(4u^2+v)^2$

14. Consider the surface given implicitly by the equation $xyz - 4y^2z^2 + \cos(xy) = 0$

    (a) Use the Chain Rule to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

    Notice that $F_x = yz - y\sin(xy)$, $F_y = xz - 8yz^2 - x\sin(xy)$, and $F_z = xy - 8y^2z$

    Recall that $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{yz - y\sin(xy)}{xy - 8y^2z}$

    Similarly, $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{xz - 8yz^2 - x\sin(xy)}{xy - 8y^2z}$

    (b) Find an equation for the tangent plane to this surface at the point $(0, 1, \frac{1}{2})$

    From above, $\nabla\vec{F} = \langle yz - y\sin(xy), xz - 8yz^2 - x\sin(xy), xy - 8y^2z\rangle$

    Then $\nabla\vec{F}(0,1,\frac{1}{2}) = \langle \frac{1}{2} - 1\sin(0), 0 - 8(1)\left(\frac{1}{2}\right)^2 - 0\sin(0), 0 - 8(1)^2\left(\frac{1}{2}\right)\rangle = \langle \frac{1}{2}, -2, -4\rangle$

    Then the tangent plane is given by the equation: $\frac{1}{2}x - 2(y-1) - 4(z - \frac{1}{2}) = 0$ or $\frac{1}{2}x - 2y - 4z + 4 = 0$

15. Recall that when translating from rectangular to polar coordinates $r = \sqrt{x^2+y^2}$.

    (a) Show that $\dfrac{\partial r}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{r} = \cos\theta$

    Notice that we are looking at the conversion formulas from rectangular to polar coordinates as functions of two variables.

    Now, $\dfrac{\partial r}{\partial x} = \dfrac{1}{2}(x^2+y^2)^{-\frac{1}{2}}2x = \dfrac{x}{\sqrt{x^2+y^2}} = \dfrac{x}{r}$.

    Moreover, since $x = r\cos\theta$, $\frac{x}{r} = \cos\theta$. We also have that $r = \frac{x}{\cos\theta}$

    (b) Starting with $r = \dfrac{x}{\cos\theta}$, does it follow that $\dfrac{\partial r}{\partial x} = \dfrac{1}{\cos\theta}$? Why or why not?

    It is tempting to view $\cos\theta$ as constant with respect to x and to compute $\frac{\partial r}{\partial x} = \frac{1}{\cos\theta}$, but since we just showed that $\frac{\partial r}{\partial x} = \cos\theta$, so know that this can't be true. The problem is our initial assumption. The expression $\cos\theta$ actually depends on x.

16. Given that $z = f(x,y) = x^3 - 2xy$

    (a) Find the equation of the tangent plane to f at $(1, -1, 3)$.

    $f_x = 3x^2 - 2y$ and $f_y = -2x$, so $f_x(1,-1) = 5$ and $f_y(1,-1) = -2$. Also notice that $f(1,-1) = 3$

    Therefore, $\vec{n} = \langle 5, -2, -1\rangle$ and the tangent plane has equation $5(x-1) - 2(y+1) - (z-3) = 0$ or $5x - 2y - z - 4 = 0$

    (b) Find an equation for the normal line to f at $(1, -1, 3)$.

    The normal line is the line through $(1,-1,3)$ in the direction of the normal vector $\langle 5, -2, -1\rangle$

    Therefore the line has parametric equation $\ell : \begin{cases} x = 1 + 5t \\ y = -1 - 2t \quad t \in \mathbb{R} \\ z = 3 - t \end{cases}$

    (c) Use the tangent plane you found to estimate $f(1.1, -.9)$. How good is your estimate?

    From the tangent plane equation, we have $z = 5x - 2y - 4$. Therefore, $f(1.1, -.9) \approx 5(1.1) - 2(-0.9) - 4 = 5.5 + 1.8 - 4 = 3.3$

    In actuality, $f(1.1, -.9) = (1.1)^3 - 2(1.1)(-0.9) = 1.331 + 1.98 = 3.311$

17. Let $f(x,y) = \sqrt{x^2+y^2}$

    (a) Find the directional derivative of f at $(3, -4)$ in the direction of $\langle 3, -2\rangle$.

    Notice that $f_x = \dfrac{x}{\sqrt{x^2+y^2}}$ and $f_y = \dfrac{y}{\sqrt{x^2+y^2}}$.

    Therefore, at $(3,-4)$, $f_x(3,-4) = \frac{3}{\sqrt{9+16}} = \frac{3}{5}$ and $f_y(3,-4) = \frac{-4}{\sqrt{9+16}} = -\frac{4}{5}$

Thus $\nabla f(3,-4) = \langle \frac{3}{5}, -\frac{4}{5} \rangle$.
Also, given $\vec{v} = \langle 3,-2 \rangle$, the unit vector in the same direction as $\vec{v}$ is $\vec{u} = \frac{\langle 3,-2 \rangle}{\sqrt{9+4}} = \langle \frac{3}{\sqrt{13}}, \frac{-2}{\sqrt{13}} \rangle$
Thus $D_{\vec{u}} f(3,-4) = \nabla f(3,-4) \cdot \vec{u} = \langle \frac{3}{5}, -\frac{4}{5} \rangle \cdot \langle \frac{3}{\sqrt{13}}, \frac{-2}{\sqrt{13}} \rangle = \frac{9}{5\sqrt{13}} + \frac{8}{\sqrt{13}} = \frac{17}{5\sqrt{13}}$

(b) Find the magnitude and direction of the maximum rate of change of f at the point $(3,-4)$.
The direction of the maximum rate of increase is $\nabla f(3,-4) = \langle \frac{3}{5}, -\frac{4}{5} \rangle$.
The magnitude of the maximum rate of increase is $\|\nabla f(3,-4)\| = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1$.

18. Find all points at which the tangent plane to the surface $z = 2x^2 - 4xy + y^4$ is parallel to the xy-plane.
Notice that the points where the tangent plane is horizontal are precisely those where both partial derivatives are zero.
Now, $f_x = 4x - 4y$ and $f_y = -4x + 4y^3$. If $f_x = 0$ then $4x = 4y$ or $x = y$.
Then, substituting, $f_y = 0$ becomes $-4x + 4x^3 = 0$, or $4x(x^2-1) = 0$, so $x = 0$ or $x = 1$ or $x = -1$.
Then, since $x = y$ the points are $(0,0)$, $(1,1)$, and $(-1,-1)$.

19. Find ∇F at $(1,2,2)$ if $F(x,y,z) = z^2 e^{2x-y} - 4xz^2$
$F_x = 2z^2e^{2x-y} - 4z^2$, $F_y = -z^2e^{2x-y}$, and $F_z = 2ze^{2x-y} - 8xz$, so $\nabla \vec{F} = \langle 2z^2e^{2x-y} - 4z^2, -z^2e^{2x-y}, 2ze^{2x-y} - 8xz \rangle$
Hence $\nabla \vec{F}(1,2,2) = \langle 8e^0 - 16, -4e^0, 4e^0 - 16 \rangle = \langle -8, -4, -12 \rangle$

20. Let $f(x,y) = x^3 - 3xy + y^3$

(a) Find all critical points of f.
If $f_x = 3x^2 - 3y = 0$, then $y = x^2$.
If $f_y = -3x + 3y^2 = 0$, substituting gives $-3x + 3x^4 = 0$, or $3x(x^3 - 1) = 0$, so $x = 0$ or $x = 1$.
Therefore, the critical points are $(0,0)$ and $(1,1)$

(b) Classify each critical point using the Discriminant.
$f_{xx} = 6x$, $f_{yy} = 6y$, and $f_{xy} = -3$
Therefore, $D(0,0) = (0)(0) - (-3)^2 = -9$, so $(0,0)$ is a saddle point.
Similarly, $D(1,1) = (6)(6) - (-3)^2 = 36 - 9 = 27$, and $f_{xx} = 6 > 0$ so $(1,1)$ is a local minimum.

21. Let $f(x,y) = 4xy - x^4 - y^4 + 4$

(a) Find all critical points of f.
If $f_x = 4y - 4x^3 = 0$, then $4y = 4x^3$, or $y = x^3$
If $f_y = 4x - 4y^3 = 0$, substituting gives $4x - 4x^9 = 0$, or $-4x(x^8 - 1) = 0$, so $x = 0$, $x = 1$ or $x = -1$.
Therefore, the critical points are $(0,0)$, $(1,1)$ and $(-1,-1)$

(b) Classify each critical point using the Discriminant.
$f_{xx} = -12x^2$, $f_{yy} = -12y^2$, and $f_{xy} = 4$
Therefore, $D(0,0) = (0)(0) - (4)^2 = -16$, so $(0,0)$ is a saddle point.
Similarly, $D(1,1) = (-12)(-12) - (4)^2 = 144 - 16 = 128$, and $f_{xx} = -12 < 0$ so $(1,1)$ is a local maximum.
Likewise, $D(-1,-1) = (-12)(-12) - (4)^2 = 144 - 16 = 128$, and $f_{xx} = -12 < 0$ so $(-1,-1)$ is a local maximum.

22. Find the absolute extrema of $w = f(x, y) = x^2 + y^2 - 2x - 4y$ on the region bounded by $y = x$, $y = 3$, and $x = 0$

First, we find all critical points. If $f_x = 2x - 2 = 0$, then $x = 1$. If $f_y = 2y - 4 = 0$, then $y = 2$, so the only critical point is $(1, 2)$.

$f(1, 2) = 1 + 4 - 2 - 8 = -5$.

Next, we check each component of the boundary:

If $y = x$, then $f(x, x) = g(x) = 2x^2 - 6x$, so $g'(x) = 4x - 6$ which has a critical point when $x = \frac{3}{2}$, which is in our region of interest.

$f(\frac{3}{2}, \frac{3}{2}) = g(\frac{3}{2}) = 2 \cdot \frac{18}{4} - 6 \cdot \frac{3}{2} = -\frac{18}{4} = -4.5$.

If $y = 3$, then $f(x, 3) = g(x) = x^2 - 2x - 3$, so $g'(x) = 2x - 2$ which has a critical point when $x = 1$.

$f(1, 3) = g(1) = 1 - 2 - 3 = -4$.

If $x = 0$, then $f(0, y) = g(y) = y^2 - 4y$, so $g'(y) = 2y - 4$ which has a critical point when $y = 2$.

$f(0, 2) = g(2) = 4 - 8 = -4$.

Finally, we check the "corner" points: $f(0, 0) = 0$, $f(0, 3) = 0 + 9 - 0 - 12 = -3$, and $f(3, 3) = 9 + 9 - 6 - 12 = 0$

Therefore the absolute maximum is 0, which occurs at $(0, 0)$ and $(3, 3)$ and the absolute minimum is -5, which occurs at $(1, 2)$

23. Find the absolute extrema of $w = f(x, y) = x^2 + y^2$ on the region bounded by $(x - 1)^2 + y^2 = 4$

First, we find all critical points. If $f_x = 2x = 0$, then $x = 0$. If $f_y = 2y = 0$, then $y = 0$, so the only critical point is $(0, 0)$.

$f(0, 0) = 0$.

Next, we check each component of the boundary:

If $(x - 1)^2 + y^2 = 4$, then $y^2 = 4 - (x - 1)^2$, so, substituting, we have $g(x) = x^2 + 4 - (x - 1)^2 = x^2 + 4 - x^2 + 2x - 1 = 3 + 2x$.

Then $g'(x) = 2$, so there are no critical points.

Notice that in our circular region $-1 \le x \le 3$, so we can check the "endpoints" as follows:

$g(-1) = 3 - 2 = 1$ and $g(3) = 3 + 6 = 9$. Also notice that of $x = 3$, then $(3 - 1)^2 + y^2 = 4$, or $4 + y^2 = 4$, so $y = 0$.

Therefore the absolute maximum is 9, which occurs at $(3, 0)$ and the absolute minimum is 0, which occurs at $(0, 0)$