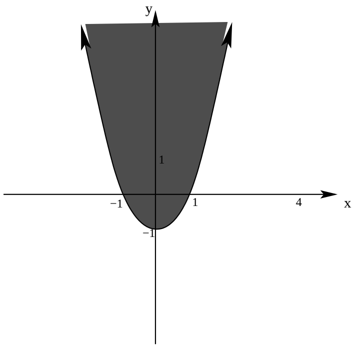
Math 323
Exam 3 **Name:**______________________________

Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

1. Let $z = f(x,y) = e^{\sqrt{y-x^2+1}}$.

   (a) (8 points) Sketch the domain of this function.

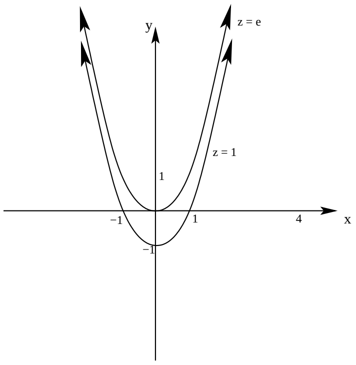
   Notice that to be in the domain of this function, we need $y - x^2 + 1 \geq 0$, or $y \geq x^2 + 1$. Thus the domain of this function is the subset of the xy-plane on or above the parabola given by the equation $y = x^2 - 1$ (See the graph below).



   (b) (8 points) Sketch contours for $z = f(x,y)$ when $z = 1$ and when $z = e$.

   When $z = 1$, $f(x,y) = e^{\sqrt{y-x^2+1}} = 1 = e^0$, so $\sqrt{y - x^2 + 1} = 0$. Thus $y - x^2 + 1 = 0^2 = 0$, or $y = x^2 - 1$.

   When $z = e$, $f(x,y) = e^{\sqrt{y-x^2+1}} = e^1$, so $\sqrt{y - x^2 + 1} = 1$. Thus $y - x^2 + 1 = 1^2 = 1$, or $y = x^2$.



2. (10 points) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + 2xy + y^2}$ does not exist.

   Computing the limit along many pairs of paths would show that this limit does not exist. Here is one possible pair:

   Along $y = 0$: $\displaystyle\lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0$ (technically, we use L'Hopital's Rule twice to evaluate this limit in one variable).

   Along $x = y$: $\displaystyle\lim_{(x,x)\to(0,0)} \frac{x^2}{x^2 + 2x^2 + x^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{4x^2} = \lim_{(x,x)\to(0,0)} \frac{1}{4} = \frac{1}{4}$.

   Since the value of the limit along these two paths does not agree, this limit does not exist.

3. Given the equation: $4\cos(xyz) + x^3y - 3yz^2 - 7 = 0$:

(a) (8 points) Use implicit differentiation to find $\frac{\partial z}{\partial x}$

Recall that $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$.

Also, $F_x = -4yz\sin(xyz) + 3x^2y$ and $F_z = -4xy\sin(xyz) - 6yz$.

Then $\frac{\partial z}{\partial x} = -\frac{-4yz\sin(xyz) + 3x^2y}{-4xy\sin(xyz) - 6yz} = \frac{-4z\sin(xyz) + 3x^2}{4x\sin(xyz) + 6z}$

(b) (8 points) Find an equation for the tangent plane to this surface at the point $(0, -1, 1)$.

We first use the gradient to find a normal vector to the tangent plane:

$\nabla F = \langle F_x, F_y, F_z \rangle = \langle -4yz\sin(xyz) + 3x^2y, -4xz\sin(xyz) + x^3 - 3z^2, -4xy\sin(xyz) - 6yz \rangle$

Then $\nabla F(0, -1, 1) = \langle -4(-1)(1)\sin(0) + 3(0)^2(-1), -4(0)(1)\sin(0) + 0^3 - 3(1)^2, -4(0)(-1)\sin(0) - 6(-1)(1) \rangle = \langle 0, -3, 6 \rangle = \vec{n}$

Using $\vec{n} = \langle 0, -3, 6 \rangle$ and the point $P(0, -1, 1)$, the desired tangent plane has the following formula:

$0(x - 0) - 3(y + 1) + 6(z - 1) = 0$. Simplifying gives $-3y - 3 + 6z - 6 = 0$ or $y - 2z = -3$

4. (10 points) Let $w = f(x, y) = 5x^2y^3$.
Find the differential dw and use it to approximate Δw as the input changes from $(1, 2)$ to $(1.1, 1.8)$

First, recall that $dw = f_x(x, y)dx + f_y(x, y)dy$.

Here, $f_x(x, y) = 10xy^2$, $f_y(x, y) = 15x^2y^2$, $dx = \Delta x = 0.1$, $dy = \Delta y = -0.2$, $x = 1$, and $x = 2$.

Then $dw = 10xy^2dx + 15x^2y^2dy = 10(1)(2)^3(0.1) + 15(1)^2(2)^2(-0.2)$

$= (80)(0.1) + (60)(-0.2) = 8 - 12 = -4$.

5. (10 points) Suppose $w = f(x, y)$ where $x = 2s^2t^2$ and $y = 4s - t$. Also suppose that $f_x(x, y) = 2xy$ and $f_y(x, y) = x^2$. Find the value of $\dfrac{\partial w}{\partial s}$ when $s = 1$ and $t = 2$.

Recall that $\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} = f_x(x, y) \cdot \frac{\partial x}{\partial s} + f_y(x, y) \cdot \frac{\partial y}{\partial s}$

Furthermore, $\frac{\partial x}{\partial s} = 4st^2$, $\frac{\partial y}{\partial s} = 4$, and we are given $f_x(x, y) = 2xy$ and $f_y(x, y) = x^2$.

Then $\frac{\partial w}{\partial s} = (2xy)(4st^2) + (x^2)(4)$. Notice that when $s = 1$ and $t - 2$, then $x = 2(1)^2(2)^2 = 8$, and $y = 4(1) = 2 = 2$.

Substituting, we have $\frac{\partial w}{\partial s} = 2(8)(2) \cdot 4(1)(2)^2 + 8^2(4) = 512 + 256 = 768$.

6. (10 points) Given that $z = f(x, y) = xe^{xy}$ find the derivative of f at the point $(1, 1)$ and in the direction of the vector $\langle -4, 3 \rangle$.

Recall that $D_{\vec{u}}f(a, b) = \nabla f(a, b) \cdot \vec{u}$. Here, $\nabla f = \langle f_x, f_y \rangle = \langle e^{xy} + xye^{xy}, x^2e^{xy} \rangle$

Then $\nabla f(1, 1) = \langle e^1 + (1)(1)e^1, (1)^2e^1 \rangle = \langle 2e, e \rangle$.

Also, $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{\langle -4, 3 \rangle}{\sqrt{16 + 9}} = \dfrac{\langle -4, 3 \rangle}{\sqrt{25}} = \langle -\dfrac{4}{5}, \dfrac{3}{5} \rangle$

Then $D_{\vec{u}}f(1, 1) = \nabla f(1, 1) \cdot \vec{u} = \langle 2e, e \rangle \cdot \langle -\frac{4}{5}, \frac{3}{5} \rangle = -\frac{8e}{5} + \frac{3e}{5} = -\frac{5e}{5} = -e$

7. (16 points) Find all the critical points of $f(x, y) = 3x - x^3 - 2y^3 + 3y^2$, and classify them using the Discriminant.

Since $f(x, y)$ is continuous (it is a polynomial in two variables), its critical points occur when $f_x = f_y = 0$. Notice that $f_x(x, y) = 3 - 3x^2$ and $f_y(x, y) = -6y^2 + 6y$.

If $f_x = 0$, then $3 - 3x^2 = 0$, so $3x^2 = 3$, or $x^2 = 1$. Thus $x = \pm 1$.

Similarly, if $f_y = 0$, then $-6y^2 + 6y = 0$, or $-6y(y - 1) = 0$. Thus $y = 0$ or $y = 1$.

Therefore, the critical values for this function are: $(1, 0), (-1, 0), (1, 1)$, and $(-1, 1)$.

Next, we classify these critical values using the discriminant. Recall that $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$.

Notice that $f_{xx} = -6x$, $f_{yy} = -12y + 6$, and $f_{xy} = 0$. Then we have the following:

$D(1, 0) = (-6)(6) - 0^2 = -36$, so a saddle point occurs at $(1, 0, f(1, 0))$.

$D(-1, 0) = (6)(6) - 0^2 = 36$, and $f_{xx}(-1, 0) = 6 > 0$ so a local minimum occurs at $(-1, 0, f(-1, 0))$.

$D(1, 1) = (-6)(-6) - 0^2 = 36$, and $f_{xx}(1, 1) = -6 < 0$ so a local maximum occurs at $(1, 1, f(1, 1))$.

$D(-1, 1) = (6)(-18) - 0^2 = -108$, so a saddle point at $(-1, 1, f(1, 1))$.

8. (16 points) Use Lagrange multipliers to find the vector in 3-space whose length is 5 and whose components have the largest possible sum.

Notice that we are trying to maximize the sum of the components of a 3D vector $\vec{v} = \langle a, b, c \rangle$. Therefore, the function we want to maximize is $f(a, b, c) = a + b + c$.

Next, we see that the length of the vector must be 5. Since the length is given by the magnitude or norm of the vector, this tells us that $\|\vec{v}\| = \sqrt{a^2 + b^2 + c^2} = 5$, or, in a much nicer form: $a^2 + b^2 + c^2 = 25$. This tells us that the contraint is given by $g(a, b, c) = a^2 + b^2 + c^2 - 25 = 0$.

Using Lagrange's method, we recall that solutions occur when $\nabla f = \lambda \nabla g$ and at points where the constraint equation $g(x, y) = 0$ is satisfied.

We see that $\nabla f = \langle f_a, f_b, f_c \rangle = \langle 1, 1, 1, \rangle$ and $\nabla g = \langle g_a, g_b, g_c \rangle = \langle 2a, 2b, 2c \rangle$

Then we have $1 = \lambda 2a$ $1 = \lambda 2b$ and $1 = \lambda 2c$. Therfore, $\frac{1}{2\lambda} = a = b = c$.

Combining this with the constraint equation $g(a, b, c) = a^2 + b^2 + c^2 - 25 = 0$ gives $a^2 + a^2 + a^2 - 25 = 0$, or $3a^2 = 25$.

Then $a^2 = \frac{25}{3}$, so $a = \pm\frac{5\sqrt{3}}{3}$.

Since the negative value cannot give a maximum, we conclude that the maximum values occurs when $a = b = c = \frac{5\sqrt{3}}{3}$. Hence the vector in 3-space whose length is 5 and whose components have the largest possible sum is $\left\langle \frac{5\sqrt{3}}{3}, \frac{5\sqrt{3}}{3}, \frac{5\sqrt{3}}{3} \right\rangle$