Math 323
Green's Theorem Examples

Recall: Green's Theorem States: Let $\mathcal{C}$ be a piecewise smooth simple closed curve, and let R be the region consisting of $\mathcal{C}$ and its interior. If M and N are continuous functions that have continuous first parial derivatives throughout an open region D containing R, then:

$$\oint_{\mathcal{C}} M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,dA$$

Similarly: working backwards: Let R be a regoin in the xy plane bounded by a piecewise smooth simple closed curve $\mathcal{C}$. Then the area A of R is given by:

$$A = \oint_{\mathcal{C}} x\,dy = -\oint_{\mathcal{C}} y\,dx = \frac{1}{2}\oint_{\mathcal{C}} x\,dy - y\,dx$$

Examples:

1. Use Green's Theorem to evaluate $\oint_{\mathcal{C}} 3y, dx + 2x^2\,dy$ where $\mathcal{C}$ is the circle $x^2 + y^2 = 4$ oriented counterclockwise.

   Notice this for this line integral, $M(x, y) = 3y$ and $N(x, y) = 2x^2$, so $M_y = 3$ and $N_x = 4x$. Using Green's Theorem, we see that

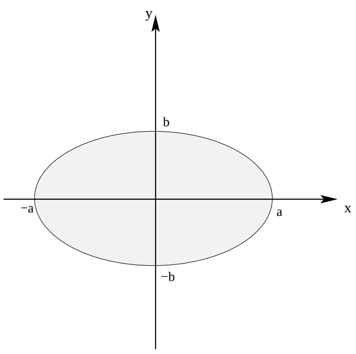
   $\oint_{\mathcal{C}} M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,dA = \iint_R 4x - 3\,dA$, where R is the circle of radius 2 centered at the origin.

   Changing to polar coordinates, we have: $\int_0^{2\pi}\int_0^2 (4r\cos\theta - 3)\ rdrd\theta = \int_0^{2\pi}\int_0^2 4r^2\cos\theta - 3r\,drd\theta = \int_0^{2\pi} \frac{4}{3}r^3\cos\theta - \frac{3}{2}r^2\Big|_0^2\,d\theta$

   $= \int_0 2\pi \frac{32}{3}\cos\theta - 6\,d\theta = \frac{32}{3}\sin\theta - 6\theta\Big|_0^{2\pi} = -12\pi$

2. Use a line integral to find a formula for the area of an ellipse of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

   An ellipse of this form is given by the following parameterization: $\mathcal{C} = \begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$ for $0 \le t \le 2\pi$



   Notice that $x'(t) = -a\sin t$ and $y'(t) = b\cos t$. It turns out that the "averaged" form for using Green's Theorem to compute area works out the smoothest, so we compute: $A = \frac{1}{2}\oint_{\mathcal{C}} x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t) - (b\sin t)(-a\sin t)\ dt = \frac{1}{2}\int_0^{2\pi} ab\cos^2 t + ab\sin^2 t\,dt$

   $= \frac{1}{2}\int_0^{2\pi} ab\,dt = \frac{1}{2}abt\Big|_0^{2\pi} = \pi ab.$