1. Mass:

(a) 2D case: Suppose that L is 2-dimensional lamina in the shape of a region R in the plane and that $\delta(x, y)$ is a function that gives the density per unit area at each point of R. Then the mass of L is given by:

$$m = \iint_R \delta(x, y) \, dA$$

(b) 3D case: Suppose that L is 3-dimensional solid in the shape of a solid region Q in space and that $\delta(x, y, z)$ is a function that gives the density per unit of volume at each point of Q. Then the mass of L is given by:

$$m = \iiint_Q \delta(x, y, z) \, dV$$

2. First Moments and the Center of Mass:

(a) 2D case: Suppose that L is 2-dimensional lamina in the shape of a region R in the plane and that $\delta(x, y)$ is a function that gives the density per unit area at each point of R. Then the first moment of L with respect to the x and y axes are:

$$M_x = \iint_R y \delta(x, y) \, dA$$
 and $M_y = \iint_R x \delta(x, y) \, dA$

The center of mass of the lamina L is $(\overline{x}, \overline{y})$, where $\overline{x} = \frac{M_y}{m}$ and $\overline{y} = \frac{M_x}{m}$

(b) 3D case: Suppose that L is 3-dimensional solid in the shape of a solid region Q in space and that $\delta(x, y, z)$ is a function that gives the density per unit of volume at each point of Q. Then the first moment of L with respect to the xy, xz, and yz planes are:

$$M_{xy} = \iiint_Q z \delta(x, y, z) \, dV, \, M_{xz} = \iiint_Q y \delta(x, y, z) \, dV, \text{ and } M_{yz} = \iiint_Q x \delta(x, y, z) \, dV$$

The center of mass of L is $(\overline{x}, \overline{y}, \overline{z})$, where $\overline{x} = \frac{M_{yz}}{m}, \, \overline{y} = \frac{M_{xz}}{m}$, and $\overline{z} = \frac{M_{xy}}{m}$

Definition: The Moment of Inertia (or the *angular mass*) of an object is a constant that reflects the inertia of a rigid object when it is revolved about some axis (or plane) of rotation. For a 2D Lamina, one can compute the moment of inertia when revolving about the x-axis (I_x) , the y-axis (I_y) , and about the origin (I_0) . For a 3D solid, one can compute the moment of inertia when rotating about the x-axis (I_x) , the y-axis (I_x) , the y-axis (I_y) , and about the z-axis (I_z) .

3. Second Moments or Moments of Inertia:

(a) 2D case: Suppose that L is 2-dimensional lamina in the shape of a region R in the plane and that $\delta(x, y)$ is a function that gives the density per unit area at each point of R. Then the moment of inertia of L with respect to the x and y axes and about the origin are:

$$I_x = \iint_R y^2 \delta(x, y) \, dA, \ I_y = \iint_R x^2 \delta(x, y) \, dA, \ \text{and} \ I_0 = \iint_R \left(x^2 + y^2\right) \delta(x, y) \, dA$$
 [We'll often change this into polar coordinates.]

(b) 3D case: Suppose that L is 3-dimensional solid in the shape of a solid region Q in space and that $\delta(x, y, z)$ is a function that gives the density per unit of volume at each point of Q. Then the moments of inertia of L with respect to the x, y, and z axes are:

$$I_x = \iiint_Q (y^2 + z^2) \,\delta(x, y, z) \, dV, \, I_y = \iiint_Q (x^2 + z^2) \,\delta(x, y, z) \, dV, \text{ and } I_z = \iiint_Q (x^2 + y^2) \,\delta(x, y, z) \, dV$$