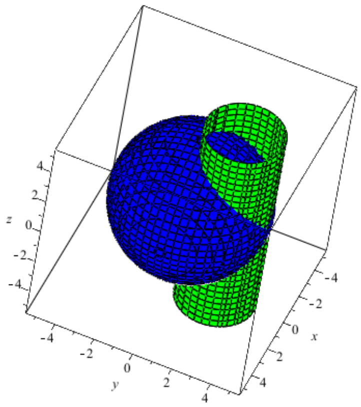
Math 323
Intersecting a Cylinder and a Sphere

Example (HW 17.3 #29): Find the volume of the region Q inside both the sphere $x^2+y^2+z^2=16$ and the cylinder $x^2+y^2=4y$. We begin by graphing this region:



Next, we begin setting up a double integral that represents the desired volume. Notice that the solid inside both the sphere and the cylinder is symmetric with respect to the xy-plane, and it is also symmetric with respect to the xz-plane. We will use this fact to simplify the integral by computing the volume of one side of the top half of the solid and multiplying this result by 4.

Evaluating the integral will be much easier if we convert to polar coordinates. Notice that the top hemisphere is given by the function $z=f(x,y)=\sqrt{16-x^2-y^2}=\sqrt{16-r^2}$. Also, using the conversion formulas $x=r\cos\theta$ and $y=r\sin\theta$, we see that the cylinder is given as follows:

$(r\cos\theta)^2+(r\sin\theta)^2=4r\sin\theta$, or $r^2\cos^2\theta+r^2\sin^2\theta=4r\sin\theta$

Then $r^2\left(\cos^2\theta+\sin^2\theta\right)=4r\sin\theta$, or $r^2=4r\sin\theta$.

Hence either $r=0$ or $r=4\sin\theta$. Since the first gives only the origin, the second must the the polar formula for the cylinder.

Therefore, $V=\iint\limits_Q f(x,y)\,dA=4\int_0^{\frac{\pi}{2}}\int_0^{4\sin\theta}\left(16-r^2\right)^{\frac{1}{2}}\,r\,dr\,d\theta=4\int_0^{\frac{\pi}{2}}-\frac{1}{3}\left(16-r^2\right)^{\frac{3}{2}}\Big|_0^{4\sin\theta}\,d\theta$

$$=-\frac{4}{3}\int_0^{\frac{\pi}{2}}\left(16-16\sin^2\theta\right)^{\frac{3}{2}}-(16)^{\frac{3}{2}}\,d\theta=-\frac{4}{3}\int_0^{\frac{\pi}{2}}\left(16\cos^2\theta\right)^{\frac{3}{2}}-(16)^{\frac{3}{2}}\,d\theta=-\frac{4}{3}\int_0^{\frac{\pi}{2}}64\cos^3\theta-64\,d\theta$$

$$=-\frac{256}{3}\int_0^{\frac{\pi}{2}}\cos\theta\left(1-\sin^2\theta\right)-1\,d\theta=-\frac{256}{3}\int_0^{\frac{\pi}{2}}\cos\theta-\cos\theta\sin^2\theta-1\,d\theta=-\frac{256}{3}\left[\sin\theta-\frac{1}{3}\sin^3\theta-\theta\Big|_0^{\frac{\pi}{2}}\right]$$

$$=-\frac{256}{3}\left[\left(1-\frac{1}{3}-\frac{\pi}{2}\right)-(0)\right]=-\frac{256}{3}\left(\frac{2}{3}-\frac{\pi}{2}\right)=-\frac{512}{9}+\frac{256\pi}{6}.$$

Note: It turns out that using symmetry to help compute the volume of this solid is not just convenient, it is absolutely necessary (pun intended). To see this, consider what happens when we compute the following:

$$2\int_0^{\pi}\int_0^{4\sin\theta}\left(16-r^2\right)^{\frac{1}{2}}\,r\,dr\,d\theta=2\int_0^{\pi}-\frac{1}{3}\left(16-r^2\right)^{\frac{3}{2}}\Big|_0^{4\sin\theta}\,d\theta$$

$$=-\frac{2}{3}\int_0^{\pi}\left(16-16\sin^2\theta\right)^{\frac{3}{2}}-(16)^{\frac{3}{2}}\,d\theta=-\frac{2}{3}\int_0^{\pi}\left(16\cos^2\theta\right)^{\frac{3}{2}}-(16)^{\frac{3}{2}}\,d\theta=-\frac{2}{3}\int_0^{\pi}64\cos^3\theta-64\,d\theta$$

$$=-\frac{128}{3}\int_0^{\pi}\cos\theta\left(1-\sin^2\theta\right)-1\,d\theta=-\frac{128}{3}\int_0^{\pi}\cos\theta-\cos\theta\sin^2\theta-1\,d\theta=-\frac{128}{3}\left[\sin\theta-\frac{1}{3}\sin^3\theta-\theta\Big|_0^{\pi}\right]$$

$$=-\frac{128}{3}\left[(0-0-\pi)-(0)\right]=\frac{128\pi}{3}.$$

Why do these two integrals give different results? Notice that there is actually a slight algebraic error in both calculations, but it is subtle. The problem is that $\left(16\cos^2\theta\right)^{\frac{3}{2}}=|4\cos\theta|^3$ (we are taking the square root of the square of a number that may or may not be positive). When we use symmetry to restrict to the first quadrant, we do not encounter negative values, so we get the correct geometric area.