Math 487
Fano's Geometry Handout

Axioms for Fano's Geometry

Undefined Terms. *point*, *line*, and *incident*.
Axiom 1. There exists at least one line.
Axiom 2. Every line has exactly three points incident to it.
Axiom 3. Not all points are incident to the same line.
Axiom 4. There is exactly one line incident with any two distinct points.
Axiom 5. There is at least one point incident with any two distinct lines.

Here are two isomorphic models for Fano's Geometry:

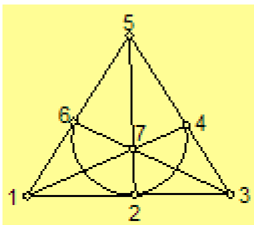


Diagram model on the left, points are defined by the seven dots and lines by the six straight segments and one curved segment. Note each line contains exactly three points.
The second model is illustrated in the table below.

points	Lines
A, B, C, D, E, F, G	*ADB, AGE, AFC, BEC, BGF, CGD, FDE*

Consider Axiom 5: In many geometries, two distinct lines intersect in exactly one point. Is the statement true with Fano's Geometry?
We investigate this question.

The statement may be reworded as
"*If two distinct lines exist, then they intersect in exactly one point.*"

By Axiom 1, we know there is at least one line.
If that line is the only line, then the statement is vacuously true.
Often this case, called the 'trivial' case, is not written when writing a proof of a conditional statement.

Let p and q be any two distinct lines.
By Axiom 5, there is a point A incident to both p and q.

We have shown there is at least one point. We need to show it is the only point.
Let's try a proof by contradiction.

Suppose there is a second point B, distinct from A, incident to both p and q.
Then by Axiom 4, p and q are the same line,
but this contradicts that p and q are distinct lines.
Thus p and q intersect in exactly one point A.
Therefore, two distinct lines intersect in exactly one point.

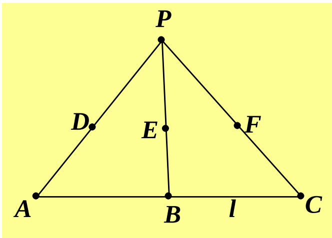
We have proven the following theorem.

Fano's Theorem 1. Two distinct lines intersect in exactly one point.

How many points does a Fano geometry have?
We investigate this question.

Investigation. By Axiom 1, there exists a line *l*.

Then by Axiom 2, there exist exactly three points *A*, *B*, *C* on line *l*.
Now by Axiom 3, there exists a point *P* not on line *l*.
Hence we have at least four distinct points *A*, *B*, *C*, and *P*.
By Axiom 4 and since *P* is not on line *l*, there are three distinct lines *AP*, *BP*, and *CP*.
And by Axiom 2, each of these lines contains a third point *D*, *E*, and *F* on *AP*, *BP*, and *CP*, respectively.



The points *A*, *B*, *C*, *D*, *E*, *F*, and *P* are distinct.
We show one case the others are left for you.
Suppose *D* = *E*. Then *AP* = *DP* = *EP* = *BP*.
This implies that *P* is on line *AB*, a contradiction.
Hence there are at least seven distinct points *A*, *B*, *C*, *D*, *E*, *F*, and *P*.

We assert that there are exactly seven distinct points.
Suppose there exists a distinct eighth point *Q*.
Note *Q* is not on *l*, since *A*, *B*, and *C* are the only points on *l*.

By Axioms 4 and 5, lines *PQ* and *l* must intersect at a point *R*.
Since *A*, *B*, and *C* are the only points on *l*, *R* must be one of *A*, *B*, or *C*.
Suppose *R* = *A*.
Since *D* is on line *AP* and *A* = *R* is on line *PQ*, we would have *R* = *A*, *D*, *P*, and *Q* collinear which contradicts Axiom 2.
The other cases for *B* or *C* are similar.
Hence there are exactly seven distinct points.

We have proven the following theorem.

Fano's Theorem 2. Fano's geometry consists of exactly seven points.