1. Give the definition of each of the following terms:

(a) A complete quadrangle

A complete quadrangle is a set of four points, no three of which are collinear, and the six lines incident with each pair of these points. The four points are called vertices and the six lines are called sides of the quadrangle.

(b) A complete quadrilateral

A complete quadrilateral is a set of four lines, no three of which are concurrent, and the six points incident with each pair of these lines. The four lines are called sides and the six points are called vertices of the quadrilateral.

(c) A perspectivity between pencils of points

A one-to-one mapping between two pencils of points is called a perspectivity if the lines incident with the corresponding points of the two pencils are concurrent. The point where the lines intersect is called the center of the perspectivity.

(d) A perspectivity between pencils of lines

A one-to-one mapping between two pencils of lines is called a perspectivity if the points of intersection of the corresponding lines of the two pencils are collinear. The line containing the points of intersection is called the axis of the perspectivity.

(e) A projectivity between pencils of points

A one-to-one mapping between two pencils of points is called a projectivity if the mapping is a composition of finitely many elementary correspondences or perspectivities.

(f) The harmonic conjugate of a point C with respect to points A and B.

Four collinear points $A, B, C, D$ form a harmonic set, denoted $H(AB, CD)$, if $A$ and $B$ are diagonal points of a quadrangle and $C$ and $D$ are on the sides determined by the third diagonal point. The point $C$ is the harmonic conjugate of $D$ with respect to $A$ and $B$.

(g) A point conic

A point conic is the set of points of intersection of corresponding lines of two projectively, but not perspectively, related pencils of lines with distinct centers.

(h) A line conic

A line conic is the set of lines that join corresponding points of two projectively, but not perspectively, related pencils of points with distinct axes.

2. State each of the following:

(a) Desargues' Theorem

If two triangles are perspective from a point, then they are also perspective from a line.

(b) The Fundamental Theorem of Projective Geometry

A projectivity between two pencils of points is uniquely determined by three pairs of corresponding points.
3. True or False

(a) In a plane projective geometry, if two triangles are perspective from a point, then they are also perspective from a line.

True. This is a consequence of Desargues’ Theorem

(b) In the Poincaré Half Plane, if two triangles are perspective from a point, then they are also perspective from a line.

False. See Homework Exercise #4.18 [Hint: pick a pair of triangles with a pair of corresponding sides that are parallel.]

(c) In a plane projective geometry, if two triangles are perspective from a line, then they are also perspective from a point.

True. This is a consequence of the dual of Desargues’ Theorem.

(d) Every point in a plane projective geometry is incident with at least 4 distinct lines.

True. This is a consequence of the dual of Theorem 4.4, which is true since Plane Projective Geometries satisfy the principle of duality.

(e) If $H(AB, CD)$ then $H(CD, BA)$.

True. This is a consequence of Theorem 4.8.

(f) If $H(AB, CD)$ and $H(AB, C'D)$ then $C = C'$

True. This is a consequence of the Fundamental Theorem 4.7.

(g) If $A, B, C$ and $A', B', C'$ are distinct elements in pencils of points with distinct axes $p$ and $p'$, there exists a perspectivity such that $ABC \sim A'B'C'$

False. Theorem 4.10 guarantees that there is a projectivity such that $ABC \wedge A'B'C'$, but this projectivity is not necessarily a perspectivity (for example, the construction we did in class to prove this theorem required two perspectivities).

4. Prove that Axiom 3 in independent of Axiom 1 and Axiom 2.

Consider the following model:

![Diagram](https://via.placeholder.com/150)

In this model, $A, B,$ and $C$ are points, and $l, m, n$ are lines. Notice that any pair of distinct points are on exactly one line [$A$ and $B$ are on $m$, $A$ and $C$ are on $l$, and $B$ and $C$ are on $n$]. Also notice that any two distinct lines are incident with at least one point [in fact, $l \cot m = A, l \cot n = C$, and $m \cot n = B$]. However, since there are only 3 points in this model, Axiom 3 is not satisfied.
5. (a) State and prove the dual of Axiom 3.

Recall Axiom 3 states: There exist at least four points, no three of which are collinear.
Then the Dual of Axiom 3 is: There exist at least four lines, no three of which are concurrent.

**Proof:** Let \( A, B, C, \) and \( D \) be four distinct points, no three of which are collinear (we know these points exist by Axiom 3). Using Axiom 1, the lines \( \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}, \overrightarrow{BC}, \overrightarrow{BD}, \) and \( \overrightarrow{CD} \) all exist. Since no three of the points \( A, B, C, \) and \( D \) are collinear, these six lines must be distinct.

Consider the four lines \( \overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \) and \( \overrightarrow{DA}. \) To show that no three of these lines are concurrent, we proceed by contradiction. Suppose not. Then three of these lines would be concurrent. For example, suppose that \( \overrightarrow{AB}, \overrightarrow{BC}, \) and \( \overrightarrow{CD} \) are concurrent. Using the Dual of Axiom 1, \( B \) is the only point of intersection of \( \overrightarrow{AB} \) and \( \overrightarrow{BC}. \) Therefore, \( B \) must be the point of concurrency for the three lines \( \overrightarrow{AB}, \overrightarrow{BC}, \) and \( \overrightarrow{CD}. \) But then \( B \) is on \( \overrightarrow{CD}. \) This contradicts our assumption that \( B, C, \) and \( D \) are noncollinear. The other cases are similar. Therefore, there exist at least four lines, no three of which are concurrent. \( \Box \)

(b) State and prove the dual of Axiom 4.

Recall that Axiom 4 states: The three diagonal points of a complete quadrangle are never collinear.
Then the Dual of Axiom 4 is: The three diagonal lines of a complete quadrilateral are never concurrent.

**Proof:** Let \( abcd \) be a complete quadrilateral (we know that such a quadrilateral exists from the Dual of Axiom 3). Let \( E = a \cdot b, F = b \cdot c, G = c \cdot d, H = a \cdot d, I = a \cdot c \) and \( J = b \cdot d. \) These points exist by Axiom 2, and are unique by the Dual of Axiom 1. Using Axiom 1, the diagonal lines \( \overrightarrow{EG}, \overrightarrow{FH}, \) and \( \overrightarrow{IJ} \) exist.

**Claim:** The diagonal lines \( \overrightarrow{EG}, \overrightarrow{FH}, \) and \( \overrightarrow{IJ} \) are not concurrent. We will prove this claim using proof by contradiction. Suppose that the lines \( \overrightarrow{EG}, \overrightarrow{FH}, \) and \( \overrightarrow{IJ} \) are concurrent. Then \( \overrightarrow{EG} \cdot \overrightarrow{FH} \) must be the point of concurrency between these lines. Therefore, the points \( I, J, \) and \( \overrightarrow{EG} \cdot \overrightarrow{FH} \) are collinear.

Since \( abcd \) is a complete quadrilateral, no three of the lines \( a = \overrightarrow{EH}, b = \overrightarrow{EF}, c = \overrightarrow{FG}, \) and \( d = \overrightarrow{GH} \) are concurrent. Thus, (using the dual of the argument in the proof of the Dual of Axiom 3) \( E, F, G, \) and \( H \) are four points, no three of which are collinear. Hence, \( EFGH \) is a complete quadrangle with diagonal points \( \overrightarrow{EF} \cdot \overrightarrow{GH} = b \cdot d = J, \overrightarrow{EG} \cdot \overrightarrow{FH} \), and \( \overrightarrow{EHFG} = a \cdot c = I. \) Hence, using Axiom 4, then the points \( I, J, \) and \( \overrightarrow{EG} \cdot \overrightarrow{FH} \) are noncollinear, which contradicts our previous assumption that they are collinear. Therefore, the diagonal lines of the complete quadrilateral \( abcd \) are not concurrent. \( \Box \)

6. (a) Prove that a complete quadrangle exists.

**Proof:** By Axiom 3, there are 4 distinct points no three of which are collinear. Call these points \( A, B, C, \) and \( D. \)
By Axiom 1, the lines \( \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}, \overrightarrow{BC}, \overrightarrow{BD}, \) and \( \overrightarrow{CD} \) all exist. We claim that these six lines are all distinct. To see this, first suppose that \( \overrightarrow{AB} = \overrightarrow{AC}. \) This would cause \( A, B, \) and \( C \) to be collinear, which contradicts our earlier assumption. The other cases are similar (note that in the case where we assume \( AB = CD \) we have that \( A, B, C, \) and \( D \) are all collinear.) Consequently, a complete quadrangle exists. \( \Box \)
(b) Draw a model for a complete quadrangle $EFGH$.

![Diagram of a complete quadrangle]

The points $E$, $F$, $G$, and $H$, along with the lines $\overrightarrow{EF}$, $\overrightarrow{EG}$, $\overrightarrow{EH}$, $\overrightarrow{FG}$, $\overrightarrow{FH}$, and $\overrightarrow{GH}$ form a complete quadrangle.

(c) Identify the pairs of opposite sides in the quadrangle $EFGH$.

There are 3 pairs of opposite sides in the quadrangle:

- $\overrightarrow{EF}$ and $\overrightarrow{GH}$
- $\overrightarrow{EH}$ and $\overrightarrow{FG}$
- $\overrightarrow{EG}$ and $\overrightarrow{FH}$

(d) Construct and identify the diagonal points of the quadrangle $EFGH$.

![Diagram with diagonal points]

Let $I = \overrightarrow{EF} \cdot \overrightarrow{GH}$
Let $J = \overrightarrow{EH} \cdot \overrightarrow{FG}$
Let $K = \overrightarrow{EG} \cdot \overrightarrow{FH}$

Then $I$, $J$, and $K$ are the diagonal points of this complete quadrangle.

7. (a) Prove that a complete quadrilateral exists.

**Proof:** By Axiom 3, there are 4 distinct points no three of which are collinear. Call these points $A$, $B$, $C$, and $D$. By Axiom 1, the lines $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{BC}$, $\overrightarrow{BD}$, and $\overrightarrow{CD}$ all exist. As in the proof of the existence of a complete quadrangle, these six lines are all distinct, otherwise, three of the original points would be collinear contrary to our previous assumption.

Consider the lines $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, and $\overrightarrow{DA}$. Using the dual of Axiom 1, let $E = \overrightarrow{AB} \cdot \overrightarrow{CD}$ and let $F = \overrightarrow{AD} \cdot \overrightarrow{BC}$. Notice that $E$ and $F$ must be distinct from $A$, $B$, $C$, and $D$, otherwise this would once again force 3 of our original points to be collinear, contrary to our previous assumption. From this, we see that no three of the lines $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, and $\overrightarrow{DA}$ are concurrent.

Hence the points $A$, $B$, $C$, $D$, $E$, and $F$ along with the lines $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, and $\overrightarrow{DA}$ form a complete quadrilateral. $\square$. 
(b) Draw a model for a complete quadrilateral $abcd$.

The lines $a$, $b$, $c$, and $d$ along with the points $J$, $U$, $S$, $T$, $I$, and $N$ form a complete quadrilateral.

(c) Identify the pairs of opposite points in the quadrilateral $abcd$.

There are three pairs of opposite points in this quadrilateral: $J$ and $I$; $U$ and $T$; $S$ and $N$.

(d) Construct and identify the diagonal lines of the quadrilateral $abcd$.

The diagonal lines in this quadrilateral are $JI$, $UT$, and $SN$.

8. (a) Construct an example of two triangles that are perspective from a point. Be sure to identify the point $O$ that the triangles are perspective from.

In the diagram above, $\triangle ABC$ and $\triangle A'B'C'$ are perspective from the point $O$.

(b) Are these two triangles also perspective from a line? If so, identify the line that the triangles are perspective from. If not, explain why they cannot be perspective from a line.

From the diagram above, if we let $\overrightarrow{AB} \cdot \overrightarrow{A'B'} = P$, $\overrightarrow{AC} \cdot \overrightarrow{A'C'} = Q$, and $\overrightarrow{BC} \cdot \overrightarrow{B'C'} = R$, notice that $R$ is incident with the line $PQ$, so $\triangle ABC$ and $\triangle A'B'C'$ are perspective from the line $PQ$.
9. Illustrate a projectivity from a pencil of lines $a, b, c$ with center $O$ to a pencil of lines $a', b', c'$ with center $O' \neq O$.

10. Prove each of the following:

(a) The dual of Desargues’ Theorem

**Dual of Desargues’ Theorem:** If two triangles are perspective from a line, then they are also perspective from a point.

**Proof:** Suppose $\triangle ABC$ and $\triangle A'B'C'$ are perspective from a line. Let $P = \overrightarrow{AB} \cdot \overrightarrow{A'B'}$, $Q = \overrightarrow{BC} \cdot \overrightarrow{B'C'}$ and $R = \overrightarrow{AC} \cdot \overrightarrow{A'C'}$. By the definition of perspectivity from a line, the points $P, Q$ and $R$ are collinear. Let $O = \overrightarrow{AA'} \cdot \overrightarrow{BB'}$. To show that $\overrightarrow{AA'}, \overrightarrow{BB'}$ and $\overrightarrow{CC'}$ are concurrent, we must show that $O$ is on the line $\overrightarrow{CC'}$.

Consider the triangles $\triangle RAA'$ and $\triangle QBB'$. Since $P, Q, R$ are collinear, $P$ is on line $\overrightarrow{QR}$. Since $P = \overrightarrow{AB} \cdot \overrightarrow{A'B'}$, $P$ is on line $\overrightarrow{AB}$ and line $\overrightarrow{A'B'}$. Hence triangles $\triangle RAA'$ and $\triangle QBB'$ are perspective from point $P$, by the definition of perspectivity from a point.

Hence by Axiom 5 (Desargues’ Theorem), triangles $\triangle RAA'$ and $\triangle QBB'$ are perspective from a line. By definition of perspectivity from a line, the points $C = \overrightarrow{RA} \cdot \overrightarrow{QB}$, $C' = \overrightarrow{RA'} \cdot \overrightarrow{QB'}$ and $O = \overrightarrow{AA'} \cdot \overrightarrow{BB'}$ are collinear. Hence $O$ is on the line $\overrightarrow{CC'}$. Therefore $\overrightarrow{AA'}, \overrightarrow{BB'}$ and $\overrightarrow{CC'}$ are concurrent. Therefore, $\triangle ABC$ and $\triangle A'B'C'$ are perspective from point $O$. $\square$.

(b) Theorem 4.6

**Theorem:** If $A, B,$ and $C$ are three distinct collinear points, then a harmonic conjugate of $C$ with respect to $A$ and $B$ exists.

**Proof:** Let $A, B,$ and $C$ be three distinct collinear points. By Axiom 3, there is a point $E$ such that $A, C,$ and $E$ are non-collinear. By Theorem 4.3, there is a point $F$ on $\overrightarrow{AE}$ that is distinct from $A$ and $E$. Let $G = \overrightarrow{CE} \cdot \overrightarrow{BF}$ and let $H = \overrightarrow{AC} \cdot \overrightarrow{BE}$.

**Claim:** The points $E, F, G,$ and $H$ and the lines $\overrightarrow{EF}, \overrightarrow{EG}, \overrightarrow{EH}, \overrightarrow{FG}, \overrightarrow{FH}$, and $\overrightarrow{GH}$ determine a complete quadrangle.
To see this, notice that the points $E, F, G$ and $H$ are distinct. $E$ and $F$ are distinct by construction. For the others, first suppose that $G = F$. Since $A$ is incident to $\overrightarrow{EF}$ and $G = F$ is incident to $\overrightarrow{CE}$, then $A, E, G = F, C$ is a collinear set, contrary to our previous assumptions. The other cases are similar.

Next, suppose that $E, F$ and $G$ are collinear. Since $G$ is incident to $\overrightarrow{BF}$, $F$ is incident to $\overrightarrow{AE}$, and $A$ is incident to $\overrightarrow{AB}$, then $A, C$, and $E$ are collinear, contrary to our previous assumptions. The other cases are similar.

This proves the claim.

Notice that $\overrightarrow{FH}$ is the remaining side of the complete quadrangle. Then if we take $D = \overrightarrow{FH} \cdot \overrightarrow{AB}$, then we have constructed the harmonic set $H(AB, CD)$. □.

(c) The Fundamental Theorem of Projective Geometry

**Theorem:** A projectivity between two pencils of points is uniquely determined by three pairs of corresponding points.

**Proof:** We must show that if $A, B, C$, and $D$ are in a pencil of points with axis $p$ and $A', B', C'$ are in a pencil of points with axis $p'$, then there exists a unique point $D'$ on $p'$ such that $ABCD \wedge A'B'C'D'$.

Assume $A, B, C$, and $D$ are in a pencil of points with axis $p$ and that $A', B'$, and $C'$ are in a pencil of points with axis $p'$. Recall that there exists a point $D'$ on $p'$ such that $ABCD \wedge A'B'C'D'$ (to find $D$, we find the image of $D$ under the first elementary correspondence, and then find the image of $d$ under the second elementary correspondence, and continue through each of the finitely many elementary correspondences in the projectivity). Suppose there is a projectivity and a point $D''$ such that $ABCD \wedge A'B'C'D''$. Since $A'B'C'D' \wedge ABCD$ and $ABCD \wedge A'B'C'D''$, we have $A'B'C'D' \wedge A'B'C''D''$. Therefore, using Axiom 6, $D' = D''$. □.

11. The frequency ratio $3 : 4 : 5$ is also equivalent to the ratio $\frac{3}{5} : \frac{15}{8} : \frac{9}{8}$, which gives the chord $G, B, D$ called the dominant of the major triad of the example above. Show $H(OG, DB)$ where $OG = \left(\frac{3}{5}\right)OC$, $OB = \left(\frac{15}{18}\right)OC$, and $OD = \left(\frac{9}{15}\right)OC$.

In the diagram above, we have constructed the harmonic set $H(OG, DB)$. 
12. Answer the following questions based on the following diagram:

(a) Find $D$, the harmonic conjugate of $C$ with respect to $A$ and $B$.

To find the harmonic conjugate of $C$ with respect to $A$ and $B$, we construct an appropriate quadrangle (one with $A$ and $B$ as diagonal points and $C$ the intersection of one of the remaining pair opposite sides) we then construct $D$ to complete the harmonic set by finding the point that the remaining opposite side intersects the line $\overrightarrow{AB}$.

(b) Pick a point $E$ not on $\overrightarrow{AB}$ and construct an elementary correspondence between the points $A, B, C, D$ and a pencil of lines with center $E$.

The diagram given above illustrates the elementary correspondence $ABCD \triangleleft abcd$.
(c) Find a line $p'$ distinct from $p = AB$ and extend the elementary correspondence you constructed in part (b) to a perspectivity between $A, B, C, D$ and corresponding points on $p'$.

The diagram given above illustrates the perspectivity $\overline{ABCD} \not\sim \overline{A'B'C'D'}$.

(d) Extend this perspectivity to a projectivity $ABC \wedge CDA$.

The diagram shown above illustrates a projectivity $ABC \wedge CDA$. 

13. Given the following projectivity:

(a) Identify each elementary correspondance in this projectivity.

The elementary correspondances are as follows:

\( ABC \sim abc \sim A'B'C' \sim a'b'c' \sim A''B''C'' \sim a''b''c'' \)

(b) Find the image of \( D \) under this projectivity.

The image of the point \( d \) under this projectivity is the line \( d'' \) as illustrated in the following diagram: