Math 487
Exam 2 - Practice Problems

1. Short Answer/Essay Questions:

   (a) Explain the difference between an axiomatic system and a model.

   Recall: An *axiomatic system* consists of undefined terms, defined terms, axioms, and theorems. Theorems are proved using only the axioms, a logical system, and previous theorems.
   A *model* of an axiomatic system is obtained by assigning meaning to the undefined terms of the axiomatic system in such a way that the axioms are true statements about the assigned concepts. Models can be either abstract or concrete.
   With this in mind, an axiomatic system is best thought of as the logical theory produced by a given set of axioms and undefined terms. A model is a setting (either abstract or concrete) where a given logical theory holds true.

   (b) Which geometric model best describes the universe we live in?

   This is an open-ended question. Answers to this question vary, and at this time, no definitive answer is known. If we are talking about the geometry of the earth, an argument could be made for the Riemann Sphere, but for smaller land areas, we often use the Euclidean Plane. What model should be used to interstellar and intergalactic travel?

   (c) Give an example of an SMSG axiom that is *not* independent of the other axioms.

   The Ruler Placement Postulate is an example of an axiom that is *not* independent of the other axioms. See Theorem 2.5 in your textbook.

   (d) Give an example of an SMSG axiom that *is* independent of the other axioms. Provide justification for your answer.

   There are many choices here. One possibility is the SAS postulate. This axiom holds in the Euclidean Plane Model and in the Poincare Half Plane Model, but it does not hold in the Taxicab Plane Model (see HW 2.49a).

   (e) How do we know that the Euclidean Parallel Postulate is Independent of the other axioms?

   Since the Euclidean Parallel Postulate holds in the Euclidean Plane Model but does not hold in the Poincare Half-Plane Model, the Euclidean Parallel Postulate must be independent of the other axioms.

   (f) Give the definition of an equivalence relation.

   **Definition:** An *equivalence relation* $\sim$ on a set $S$ that satisfies each of the following.

   - $a \sim a$ for all $a \in S$ (reflexive property)
   - For all $a, b \in S$, if $a \sim b$ then $b \sim a$ (symmetric property)
   - For all $a, b, c \in S$, if $a \sim b$ and $b \sim c$ then $a \sim c$ (transitive property)

   (g) Give the definition of an angle bisector.

   **Definition:** The *bisector* of an angle $\angle ABC$ is a ray $\overrightarrow{BD}$ where $D$ is in the interior of $\angle ABC$, and $\angle ABD \cong \angle DBC$.

2. For each of the following, provide an example of a model where the statement is true and an example of a model where the statement is false.

   (a) There is a unique line between any pair of points.

   This statement is true in the Euclidean Plane, the Taxicab Plane, the Max-Distance Plane, the Missing Strip Plane, the Modified Riemann Sphere, and the Poincare Half-Plane model.

   This statement is false in the Riemann Sphere and in some Discrete Geometries.

(b) The ruler postulate holds.

This statement is true in the Euclidean Plane, the Taxicab Plane, the Max-Distance Plane, the Missing Strip Plane, and the Poincare Half-Plane model.

This statement is false in the Riemann Sphere, the Modified Riemann Sphere, and in Discrete Geometries.

(c) The SAS postulate holds.

This statement is true in the Euclidean Plane and Poincare Half-Plane models.

This statement is false in the Taxicab Plane and in the Max-Distance Plane models (any others?)

(d) the AAA congruence theorem is true.

This statement is true in the Poincare Half-Plane model.

This statement is false in the the Euclidean Plane model.

(e) The Plane Separation Postulate holds.

This statement is true in the Euclidean Plane, the Taxicab Plane, the Max-Distance Plane, and the Poincare Half-Plane model.

This statement is false in the Missing Strip Plane.

(f) Rectangles exist.

This statement is true in the the Euclidean Plane model.

This statement is false in the Poincare Half-Plane model and in the Riemann Sphere model.

(g) Lines have infinite length.

This statement is true in the Euclidean Plane, the Taxicab Plane, the Max-Distance Plane, the Missing Strip Plane, and the Poincare Half-Plane model.

This statement is false in the Riemann Sphere, the Modified Riemann Sphere, and in Discrete Geometries.

(h) the sum of the measures of the angles in any triangle is 180

This statement is true in the the Euclidean Plane model.

This statement is false in the Poincare Half-Plane model and in the Riemann Sphere model.

3. For each statement, determine whether the statement is true or false. Then briefly justify your answer. Assume each statement is being made about a neutral geometry (i.e. every postulates except the Euclidean parallel postulate holds).

(a) Parallel lines exist.

True.

By Postulate 5a, there are at least 3 non-collinear points. By Postulate 1, given any two points, there is exactly one line that contains them. Therefore, a lines exist. Hence there is a line $\ell$, and there is a point not on that line. Let $A$ and $B$ be points on $\ell$. By the Postulate 12, there is a ray $\overrightarrow{AP}$ in one half plane of $\overleftrightarrow{AB}$ such that $m(\angle BAP) = 90$. Similarly, if we consider the line $\overleftrightarrow{AP}$, again by Postulate 12, there is a ray $\overrightarrow{PD}$ with $D$ in the same half plane as $B$ such that $m(\angle APD) = 90$. Thus both $\overleftrightarrow{AB}$ and $\overleftrightarrow{PD}$ are perpendicular with $\overleftrightarrow{AP}$. Hence, by Theorem 2.13, $\overleftrightarrow{AB}$ and $\overleftrightarrow{PD}$ are parallel.

(b) If a line $\ell$ is perpendicular with distinct lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, then the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel.

True.

This is just a restatement of Theorem 2.13, which you proved as part of HW Assignment #5.

(c) Given and 3 distinct collinear points, exactly one is between the other two.

True.

Let $A$, $B$, and $C$ be 3 distinct collinear points. Let $\ell$ be the line containing these three points. Using the Ruler Postulate, let $f$ be a ruler for $\ell$, and let $f(A) = a$, $f(B) = b$, and $f(C) = c$. WLOG, assume $a < b < c$ (since the three points are distinct, these coordinates must be distinct). Then $d(A, B) = b - a$, $d(B, C) = c - b$, and $d(A, C) = c - a$. Also, $d(A, B) + d(B, C) = (b - a) + (c - b) = c - a = d(A, C)$. Hence $A - B - C$.

(d) If $m(\angle BAC) = m(\angle BAD) + m(\angle DAC)$, then $D \in int(\angle BAC)$.

Suppose $m(\angle BAC) = m(\angle BAD) + m(\angle DAC)$, but $D \notin int(\angle BAC)$.
Case 1: Suppose $D$ is on the same side of $\overleftrightarrow{AB}$ as $C$. Then $B$ and $D$ must be on opposite sides of $\overleftrightarrow{AC}$, and by the contrapositive of Theorem 2.7, $m(\angle BAD) \geq m(\angle BAC)$. This is a contradiction of our assumption that $m(\angle BAC) = m(\angle BAD) + m(\angle DAC)$ and the fact that all angle measures are positive.
Case 2: Suppose $D$ is on the opposite side of $\overleftrightarrow{AB}$ as $C$. By the Plane Separation Postulate, $\overline{CD}$ must intersect $\overleftrightarrow{AB}$ at a point $E$. Since $\angle DAC$ is an angle, $A \neq E$. Then either $B \in int\angle DAC$ or $E \in \int \angle DAC$. In the first case, we have $m(\angle BAC) < m(\angle BAD)$. In the second, we have $m(\angle BAD) + m(\angle DAC) > 180$.
Therefore, we must have $D \in int(\angle BAC)$.

4. Determine whether or not the Missing Strip Plane satisfies the Euclidean Parallel Postulate.

**Claim:** The Missing Strip Plane satisfies the Euclidean Parallel Postulate. To see this consider the following. Let $\ell$ be the line given by $\{(x, y) \mid y = 2, x < 0 \text{ or } x \geq 1\}$. Let $P$ be the point $(-1, 3)$. Then the lines $\ell_1 = \{(x, y) \mid y = 3, x < 0 \text{ or } x \geq 1\}$ and $\ell_2 = \{(x, y) \mid y = -x + 2, x < 0 \text{ or } x \geq 1\}$ both contain $P$ and are both parallel with the line $\ell$.

5. Prove that a line segment has a unique midpoint.

Let $\overline{AB}$ be a line segment, and suppose $d(A, B) = r$. Applying the ruler placement postulate to the line $\overleftrightarrow{AB}$, there is a ruler $f$ such that $f(A) = 0$ and $f(B) > 0$. Then $f(B) = r$. Let $C$ be the unique point satisfying $f(C) = \frac{r}{2}$ (note that there is such a point since $f$ is onto, and there is only one such point since $f$ is 1-1).
By property iii of Postulate 3, $d(A, C) = \frac{r}{2} - 0 = \frac{r}{2}$ and $d(C, B) = r - \frac{r}{2} = \frac{r}{2}$. Then $d(A, C) + d(C, B) = \frac{r}{2} + \frac{r}{2} = r = d(A, B)$, so $A - C - B$, and $C$ is a midpoint of the segment $\overline{AB}$.

To see that $C$ is unique, suppose that $M$ is a midpoint of $\overline{AB}$. Then $M$ is on $\overleftrightarrow{AB}$ and $AM = MB$, so $AM - MB = 0$. Suppose $M - A - B$. Then $MB = MA + AB = MA + r$, so $MB - MA = r > 0$, which is a contradiction. Similarly, if $A - B - M$, then $AM = AB + BM = r + BM$, so $AM - BM = r > 0$ which is a contradiction. Clearly, we cannot have $M = A$ or $M = B$, since then either $AM = 0$ and $MB = r$, or $BM = 0$ and $AM = r$, which both lead to contradictions.

Therefore, we must have $A - M - B$. Thus $AM = MB$ and $AM + MB = r$. Hence $AM = MB = \frac{r}{2}$, so, using the same ruler as above, $M$ has coordinate $\frac{r}{2}$. Thus, since $f$ is 1-1, $M = C$ and hence $C$ is unique.

6. Prove that triangle congruence is an equivalence relation.

Recall that two triangles are congruent provided all three or their corresponding sides are congruent and all three of their corresponding angles are congruent.

In HW #2.27, we proved that segment congruence is an equivalence relation. A similar proof shows that angle congruence is an equivalence relation:

- Let $\angle A$ be an angle. If $m(\angle A) = r$, then $m(\angle A) = m(\angle A)$, so $\angle A \cong \angle A$. Hence angle congruence is reflexive.

- Let $\angle A$ and $\angle B$ be angles. If $m(\angle A) \cong \angle B$, then $m(\angle A) = m(\angle B)$. But then $m(\angle B) = m(\angle A)$, so $\angle B \cong \angle A$. Hence angle congruence is symmetric.
- Let $\angle A$, $\angle B$ and $\angle C$ be angles. If $m(\angle A) \cong \angle B$ and $m(\angle B) \cong \angle C$, then $m(\angle A) = m(\angle B)$ and $m(\angle B) = m(\angle C)$. But then $m(\angle A) = m(\angle C)$, so $\angle A \cong \angle C$. Hence angle congruence is transitive.

Therefore, since both segment congruence and angle congruence are equivalence relations, so is triangle congruence.

7. Prove that the Missing Strip Place does not satisfy Pasch's Postulate.

   **Recall:** Pasch's Postulate

   Given any line $\ell$ and any triangle $\triangle ABC$, and any point $D$ on $\ell$ such that $A - D - B$, either $\ell$ intersects $\overline{AC}$ or $\ell$ intersects $\overline{BC}$

   The following gives a counterexample to Pasch's Postulate in the Missing Strip Plane. Let $A = (-4, 0)$, $B = (2, 0)$, $C = (-4, 6)$, $D = (-2, 0)$, and $\ell$ the line given by intersecting the Euclidean line $y = x + 2$ with the Missing Strip Plane. Then $\ell$ does not intersects neither $\overline{AC}$ nor $\overline{BC}$

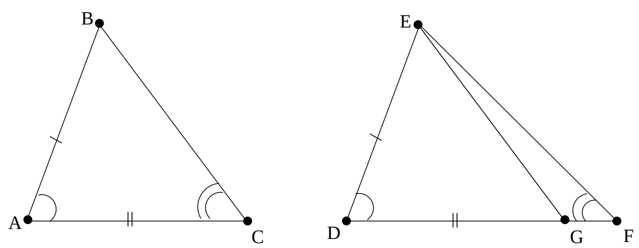
   The details are left to the reader.

8. State and prove the AAS congruence theorem.

   **Theorem:** (AAS) Let $\triangle ABC$ and $\triangle DEF$ be triangles satisfying $\angle A \cong \angle D$, $\angle C \cong \angle F$, and $\overline{AB} \cong \overline{DE}$. Then $\triangle ABC \cong \triangle DEF$

   **Proof:** We will proceed using proof by contradiction. Suppose $AC \neq DF$. WLOG, assume that $AC > DF$. Let $G$ be the point on $\overleftrightarrow{DF}$ such that $D - G - F$ and $DG = AC$.

   Consider the triangles $\triangle ABC$ and $\triangle DGE$.




   Since $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DG}$, and $\angle A \cong \angle D$, by the $SAS$ postulate, $\triangle ACB \cong \triangle DGE$. Then $\angle ACB \cong \angle DGE$. Also notice $\angle DGE$ is external to triangle $\triangle EGF$ and $\angle GFE = \angle DFE$ is a remote interior angle. But $\angle ACB \cong \angle DFE$, so $\angle DGE \cong \angle DFE$. This contradicts the conclusion of the External Angle Theorem, which requires $m(\angle DGE) > m(\angle DFE)$.

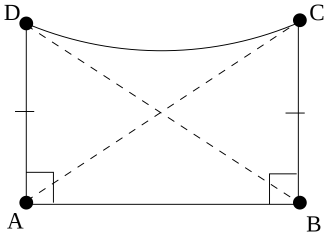
   Thus we must have $\overline{AC} \cong \overline{DF}$, hence, using the $SAS$ Postulate, $\triangle ACB \cong \triangle DFE$.

9. Prove that there are at least two lines that are parallel to each other.

   Whoops, this is the same as 3(a) above. I forgot to delete this question in the handout.

10. Prove that the **diagonals** (sorry, this was a typo – I proved the other one in class!) of a Saccheri quadrilateral are congruent.

    **Recall:** A Saccheri Quadrilateral is a quadrilateral $\square ABCD$ where $\angle BAD$ and $\angle ABC$ are right angles and $\overline{AD} \cong \overline{BC}$. Here, $\overline{AB}$ is called the *base* and $\overline{CD}$ is called the *summit*.




Since $\overline{AB} \cong \overline{AB}$, we may apply the $SAS$ Postulate to conclude that $\triangle DAB \cong \triangle CBA$. Therefore, the see that corresponding diagonal sides satisfy $\overline{AC} \cong \overline{DB}$.

11. Use the previous results along with the Theorem 2.18 and Euclid's 5th Postulate to prove that rectangles exist in a Euclidean geometry.

    Recall that from Postulate 5a and Postulate 1, there is a line $\overleftrightarrow{AB}$ and at least one point not on this line. Using the Distance Postulate, let $r = d(A,B)$. Let $H$ be one of the half planes bordering the line $\overleftrightarrow{AB}$. Applying the Angle Construction Postulate, there is exactly one ray $\overrightarrow{BP}$ in $H$ with $m(\angle ABP) = 90$. Similarly, there is exactly one ray $\overrightarrow{AQ}$ in $H$ with $m(\angle BAQ) = 90$. Applying the Ruler Postulate, let $S$ be the point on $\overrightarrow{AQ}$ with $d(A,S) = r$. Similarly, let $T$ be the point on $\overrightarrow{BP}$ with $d(B,T) = r$. Then $\square ABTS$ is a Saccheri Quadrilateral. By Theorem 2.17, $\overleftrightarrow{AB}$ and $\overleftrightarrow{ST}$ are both perpendicular to the segment joining their midpoints. Then, by Theorem 2.13, $\overleftrightarrow{AB}$ and $\overleftrightarrow{ST}$ are parallel. Similarly, since $\overleftrightarrow{AS}$ and $\overleftrightarrow{BT}$ are both perpendicular to the segment $\overline{AB}$, then, again by Theorem 2.13, $\overleftrightarrow{AS}$ and $\overleftrightarrow{BT}$ are parallel. Thus $\square ABTS$ is a parallelogram [In fact, it is s rhombus. We could have used Theorem 2.20, but I wanted to show you all the details of the proof of Theorem 2.20]. Finally, (and this is the part that is not true in general for a neutral geometry) Suppose that $\angle AST$ is not a right angle. Then, by Euclid's 5th Postulate, the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{ST}$ intersect. This is a contradiction. Hence $\angle AST$ must be a right angle. Similarly, $\angle BTS$ must also be a right angle. Thus $\square ABTS$ is a rectangle (in fact it is a square).

12. Prove each of the following Euclidean Propositions:

    (a) If $A$ and $D$ are points on the same side of a line $\overleftrightarrow{BC}$ and the line $\overleftrightarrow{BA}$ is parallel to the line $\overleftrightarrow{CD}$ then $m(\angle ABC) + m(\angle BCD) = 180$

    To obtain a contradiction, suppose that $\overleftrightarrow{CD}$ then $m(\angle ABC) + m(\angle BCD) \neq 180$.

    Case 1: Suppose $m(\angle ABC) + m(\angle BCD) < 180$.

    Then, applying Euclid's 5th Postulate, the lines $\overleftrightarrow{BA}$ and $\overleftrightarrow{CD}$ meet at a point $P$ on the same side of $\overleftrightarrow{BC}$ as $A$ and $D$. But $\overleftrightarrow{BA}$ and $\overleftrightarrow{CD}$ are parallel, so this is a contradiction.

    Case 2: Suppose $m(\angle ABC) + m(\angle BCD) > 180$.

    Let $E$ be a point on $\overleftrightarrow{AB}$ such that $A - B - E$. Similarly, let $F$ be a point on $\overleftrightarrow{CD}$ such that $D - C - F$. Using the supplement postulate and the fact that $\angle ABC$ and $\angle CBE$ form a linear pair and that $\angle DCB$ and $\angle BCF$ also form a linear pair:
    $m(\angle ABC) + m(\angle CBE) = 180$, $m(\angle DCB) + m(\angle BCF) = 180$, and by assumption, $m(\angle ABC) + m(\angle BCD) > 180$.
    Hence $m(\angle CBE) + m(\angle BCF) < 180$.
    Again applying Euclid's 5th Postulate, the lines $\overleftrightarrow{BA}$ and $\overleftrightarrow{CD}$ meet at a point $P$ on the same side of $\overleftrightarrow{BC}$ as $E$ and $F$. But $\overleftrightarrow{BA}$ and $\overrightarrow{CD}$ are parallel, so this is again a contradiction.

    Thus $m(\angle ABC) + m(\angle BCD) = 180$. $\square$.

    (b) Every distinct pair of parallel lines have a common perpendicular.

    Suppose $\overleftrightarrow{AB}$ is parallel to the line $\overleftrightarrow{CD}$ (and suppose these lines are distinct). Let $E$ be a point in the opposite half plane of $\overleftrightarrow{AB}$ as the point $C$ (since $\overleftrightarrow{AB}$ is parallel to the line $\overleftrightarrow{CD}$, $C$ is not on the line $\overleftrightarrow{AB}$). By Theorem

2.12, there is a unique perpendicular to $\overleftrightarrow{CD}$ through the point $E$. Let $F$ be the point of intersection between this perpendicular and the line $\overleftrightarrow{CD}$. Since $E$ and $C$ are in opposite half planes, by the Plane Separation Axiom, $\overline{EF}$ intersects $\overleftrightarrow{AB}$ at some point $G$.

WLOG, assume $A-G-B$ and $C-F-D$. Notice that we also have $E-G-F$ and $m(\angle GFD) = 90$.

Applying 12(a) above to the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{EF}$, we have that $m(\angle GFD) + m(\angle BGF) = 180$. Hence $m(\angle BGF) = 90$, and the line $\overleftrightarrow{EF}$ is perpendicular to both $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

(c) If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.

Suppose that we are given lines $\overleftrightarrow{AB}$, $\overleftrightarrow{DE}$, and $\overleftrightarrow{BE}$ such that $A-B-C$, $D-E-F$, and $G-B-E-H$ where $A$ and $D$ are on the same side of the line $\overleftrightarrow{BE}$. Further suppose that $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ are parallel.

Applying 12(a) above to the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{DE}$, and $\overleftrightarrow{BE}$, we have that $m(\angle CBE) + m(\angle BEF) = 180$.

However, since $\angle ABE$ and $\angle CBE$ form a linear pair, by the Supplement Postulate, $m(\angle ABE) + m(\angle CBE) = 180$. Therefore, $m(\angle ABE) = m(\angle BEF)$.

Similarly, since $\angle DEB$ and $\angle BEF$ form a linear pair, by the Supplement Postulate, $m(\angle DEB) + m(\angle BEF) = 180$. Therefore, $m(\angle CBE) = m(\angle DEB)$.

This both pairs of alternate interior angles are congruent.

(d) The sum of the measures of any triangle is 180.

Consider a triangle $\triangle ABC$. Let $\overleftrightarrow{CE}$ be the unique line through $C$ parallel to the line $\overleftrightarrow{AB}$. Let $F$ be a point such that $F-C-E$ and $G$ and $H$ be points such that $G-A-B-H$. Then, applying 12(c) to the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CE}$ and the transversal $\overleftrightarrow{AC}$, $m(\angle ACF) = m(\angle CAB)$.

Similarly, applying 12(c) to the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CE}$ and the transversal $\overleftrightarrow{CB}$, $m(\angle ECB) = m(\angle CBA)$.

Notice that $\angle FCA$ and $\angle ECA$ form a linear pair, so by the Supplement Postulate, $m(\angle FCA)+m(\angle ECA) = 180$.

Notice that since $A$ and $B$ are on the same side of $\overleftrightarrow{CE}$ and we have $A-B-H$, then $B \in int(\angle FCA)$. Thus $m(\angle FCA) = m(\angle ACB) + m(\angle FCB)$.

Therefore, $m(\angle ECA) + m(\angle ACB) + m(\angle FCB) = 180$.

But then $m(\angle CAB) + m(\angle ACB) + m(\angle CBA) = 180$.

Hence the sum of the angles of $\triangle ABC$ is 180.

13. Provide the justification for each step in the following 2-column proof (you may assume all SMSG axioms except the Parallel Postulate).

**Theorem (Exterior Angle Theorem):** Any exterior angle of a triangle $\triangle ABC$ is greater than either of its remote interior angles.

**Proof:** Given $\triangle ABC$, let $D$ be a point such that $A-C-D$ (i.e. $\angle BCD$ is an exterior angle of $\triangle ABC$).

| | |
|---|---|
| 1. Let $M$ be the midpoint of segment $\overline{BC}$. | Review Exercise 6 (HW 2.24). |
| 2. Then $B - M - C$ and $\overline{BM} \cong \overline{MC}$. | Definition of the midpoint of a segment and the definition of betweenness. |
| 3. There is a point $E$ on ray $\overrightarrow{AM}$ such that $A - M - E$ and $ME = MA$. | The Ruler Postulate, Rev. Exercise 6 (HW 2.24). |
| 4. $\overline{ME} \cong \overline{MA}$ | Definition of segment congruence. |
| 5. $\angle AMB$ and $\angle EMC$ are vertical angles. | Definition of Vertical Angles. |
| 6. $\angle AMB \cong \angle EMC$ | Theorem 2.8 |
| 7. $\triangle AMB \cong \triangle EMC$ | SAS Postulate. |
| 8. $\angle ABC = \angle ABM \cong \angle ECM = \angle ECB$ | Definition of triangle congruence, renaming. |
| 9. $m(\angle ABC) = m(\angle ECB)$ | Definition of angle congruence. |
| 10. $E$ and $D$ are on the same side of line $\overleftrightarrow{BC}$ | $A - M - E$, $A - C - D$, and the Plane Separation Axiom. |
| 11. $B$, $M$ and $E$ are on the same side of line $\overleftrightarrow{CD}$ | $A - M - E$, $C - M - B$, and the Plane Separation Axiom. |
| 12. $E \in int(\angle BCD)$ | Definition of the interior of an angle. |
| 13. $m(\angle BCE) + m(\angle ECD) = m(\angle BCD)$ | Angle Addition Postulate. |
| 14. $m(\angle BCD) > m(\angle BCE) = m(\angle ABC)$ | Angle Measurement Postulate, Subtraction, Properties if equality. |

The proof of the case for the other remote interior angle is similar $\square$.

14. Proof of Theorem 2.15: If there is a transversal to two distinct lines with alternate interior angles congruent, then the two lines are parallel.

**Proof:** Assume that $\overleftrightarrow{AB}$, $\overleftrightarrow{DE}$, and $\overleftrightarrow{BE}$ are distinct with $A - B - C$, $D - E - F$, and $G - B - E - H$. Further assume that $A$ and $D$ are on the same side of the line $\overleftrightarrow{BE}$. We will proceed with proof by contraposition. Assume that $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ are **not** parallel.

| | |
|---|---|
| 1. There is a point $P$ on both $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ | negation of the definition of parallel. |
| 2. WLOG, assume $P$ is on the same side of $\overleftrightarrow{BE}$ as $A$ | we may relabel if necessary. |
| 3. Consider $\triangle PBE$ | Step 1, definition of triangle. |
| 4. $\angle BEF$ and $\angle CBE$ are exterior angles of $\triangle PBE$ | Definition of an exterior angle. |
| 5. $m(\angle PBE) < m(\angle BEF)$ and $m(\angle PEB) < m(\angle CBE)$ | The External Angle Theorem |
| 6. $m(\angle ABE) < m(\angle BEF)$ and $m(\angle DEB) < m(\angle CBE)$ | renaming angles. |
| 7. $\angle ABE \ncong \angle BEF$ and $\angle DEB \ncong \angle CBE$ | Definition or angle congruence. |

We have shown that if two lines are not parallel, then neither pair of alternate interior angles formed by any transversal are congruent. This completes the proof by contraposition.