Math 487
Section 1.2
Fano's Geometry Theorem 2

Theorem 2: Fano's Geometry has exactly 7 points.

Proof:

By Axiom 1, there is a line ℓ_1 in this geometry. Using Axiom 2, ℓ_1 has exactly three distinct points incident to it. Call these points A, B, and C. By Axiom 3, there must be a fourth distinct point D not on ℓ_1. Using Axiom 4, there must be a line ℓ_2 incident with A and D. Similarly, there must be a line ℓ_3 incident with B and D, and a line ℓ_4 incident with C and D.

Notice that by Theorem 1, these lines must all be distinct. Since D is not on ℓ_1, then ℓ_1 is distinct from the other 3 lines. If two of the other three lines are not distinct, for example, if $\ell_2 = \ell_3$, then A, B, D would all be on this line. But then both A and B are in the intersection of ℓ_1 with this line, contradicting Theorem 1, which states that two distinct lines intersect in exactly one point.

Next, using Axiom 2, the lines ℓ_2, ℓ_3, and ℓ_4 each have a third point. Call these points E, F, and G respectively. Notice that these points must be distinct, or we would once again contradict Theorem 1. For example, if $E = F$, then ℓ_2 and ℓ_3 would intersect in both D and $E = F$ (the other cases are similar). Thus we have 7 distinct points: A, B, C, D, E, F, G.

To complete the proof, we must show that there are no other points. We proceed using proof by contradiction. Suppose there is an eighth point Q distinct from the previous 7 points. Notice that Q is not on ℓ_1, since A, B, C are the only points on ℓ_1. Similarly, Q cannot be on ℓ_2, ℓ_3, or ℓ_4, since these lines also already have three distinct points incident with them. By Axiom 4, there is a line ℓ_5 incident with D and Q. By Axiom 5, there is a point R that is incident with both ℓ_1 and ℓ_5. Since ℓ_1 is only incident with the points A, B, C, R must be one of these three points.

If $R = A$, then ℓ_5 is incident with both A and D, so by Theorem 1, $\ell_2 = \ell_5$ and so $Q = E$.
If $R = B$, then ℓ_5 is incident with both B and D, so by Theorem 1, $\ell_3 = \ell_5$ and so $Q = F$.
If $R = C$, then ℓ_5 is incident with both C and D, so by Theorem 1, $\ell_4 = \ell_5$ and so $Q = G$.

Since all of these cases lead to a contradiction, our assumption that there is an eighth distinct point must be false. Hence there are exactly 7 distinct points in Fano's Geometry. $\square$