Math 102
Expected Value

Example 1: Suppose that I have a bag containing 4 red chips, 4 white chips, and 2 blue chips.

1. What is the probability of drawing two red chips if I draw them one at a time *without* replacement?

2. What is the probability of drawing two white chips if I draw them one at a time *without* replacement?

3. What is the probability of drawing two blue chips if I draw them one at a time *without* replacement?

Example 2: Now, suppose I offer you the opportunity to play the following game: You pay me \$1 for the right to draw two chips out of the bag described in the example above. If the two chips you draw are the same color, you win \$3 (your original dollar plus \$2 more). If the two chips are different colors, I get to keep your dollar. Is it worthwhile for you to play this game?

How would we measure this?

Using the probabilities we computed above, $P(Win) = P(RR) + P(WW) + P(BB) = \_\_\_ + \_\_\_ + \_\_\_ = \_\_\_$

$P(Lose) = 1 - P(Win) = \_\_\_$

Then your expected return for playing this game is: $P(Win) \cdot (\$2) + P(Lose) \cdot (-\$1) = \_\_\_ + \_\_\_ = \_\_\_$

From this, we conclude that it (is, is not) worth playing this game.

Example 3: Now, suppose I change the rules of the game in order to make it a better deal for you to play the game.

I decide to change them so that: You pay me \$1 for the right to draw two chips out of the bag described in the original example above. If the two chips you draw are both red or is both are white, you win \$3 (your original dollar plus two more). If both of the chips are blue, you win \$5 (your original dollar plus \$4 more). If the two chips are different colors, I get to keep your dollar.

Is it worthwhile for you to play this improved version of the game?

Using the probabilities we computed above, your expected return for playing this game is:

$P(RR) \cdot (\$2) + P(WW) \cdot (\$2) + P(BB) \cdot (\$4) + P(Lose) \cdot (-\$1) = \_\_\_ + \_\_\_ + \_\_\_$ - $\_\_\_ = \_\_\_$

Definition: We can formalize this process for measuring the "worth" of an experiment whose outcomes offer different rates of return by defining the **expected value** of such an experiment. That is, given an experiment with n possible outcomes, if $P_1, P_2, ..., P_n$ are the probabilities of each of the outcomes and if $V_1, V_2, ..., V_n$ are the **values** of each outcome, then the **expected value** of the experiment is:
Expected Value $= (P_1 \cdot V_1) + (P_2 \cdot V_2) + ... + (P_n \cdot V_n)$

Definition: We say that an experiment (or "game") is **fair** if expected value of the experiment (or "game") is zero. That is, if, *on average*, you expect wins and losses to balance out to zero.
Examples:

1. In the chip drawing game above, suppose that the prize for drawing two red chips and that for drawing two white chips remains at \$3. What would the prize for drawing two blue chips need to be in order to make the game **fair**?

2. In the game of roulette, a wheel with 38 spaces all of equal size and weighting is spun. A ball is dropped onto the wheel and bounces around until it comes to rest on one of the 38 spaces. On most wheels, the spaces are labeled with the numbers 1 through 36, plus 0 and 00. Half of the numbers 1-36 are "red" and the other half are "black", while 0 and 00 are "green". You can place bets on the wheel in several ways:

   (a) The first way to bet is to pay \$1 and then pick one of the colors "red" or "black" before the wheel is spun. If your choice of color matches the result, then you win \$1 (in addition to your \$1 bet). If you are wrong, you lose your dollar bet. Find the expected value of this bet.

   (b) Another way to bet is to pay \$1 and then pick one of the numbers before the wheel is spun. If your choice of number matches the result, then you win \$35 (in addition to your \$1 bet). If you are wrong, you lose your dollar bet. Find the expected value of this bet.

3. In the Minnesota lottery "Daily 3" game, the digits 0-9 are used, and 3 of these digits are selected one at a time *with replacement* (so repetition is allowed). There are several ways to play:

   (a) The first way to bet is to pay \$1 and then pick a single digit. If your choice of digit matches the *first* number drawn, then you win \$5 (your \$1 bet plus \$4 more). If you are wrong, you lose your dollar bet. Find the expected value of this bet.

   (b) Another way to bet is to pay \$1 and then pick *two* digits, in order. If your choice of digits matches the first two numbers drawn, in order, then you win \$50 (your \$1 bet plus \$49 more). If you are wrong, you lose your dollar bet. Find the expected value of this bet.

   (c) The final way to bet is to pay \$1 and then pick *three* digits, in order. If your choice of digits matches all three numbers drawn, in order, then you win \$500 (your \$1 bet plus \$499 more). If you are wrong, you lose your dollar bet. Find the expected value of this bet.