Math 102

Permutations and Combinations Handout

Preliminary Examples:

1. Suppose that a well stocked vending machine sells 5 different types of candy bars. How many different ways can three different people purchase a single candy bar from the machine?

2. Suppose that I bring 5 different candy bars to class. Then, I pick *three* of you and let you come up, one at a time, and select *one* of the candy bars. How many ways could the three students chosen select their candy bars?

3. Suppose that I bring 5 different candy bars to class. Then, I pick *one* of you and let you come up and select *two* of the candy bars. How many ways could the student chosen select two candy bars?

Discussion: Looking at these three examples and the methods that we used to count them, there were two primary questions that we needed to address.

Question 1: Is repetition allowed?

In a counting situation, we need to be able to recognize whether or not the objects being selected get "used up"? That is, can the same object be selected more than once?

In our first example, since the vending machine was well stocked, more than one person could choose the same type of candy bar, so choices were not used up. In the next two, since there was only one candy bar of each type, repetition was not allowed.

Question 2: Does order matter?

In a counting situation, we need to be able to determine whether or not the order in which selections are made is important or not. That is, would the same choices made in a different order be considered the same outcome or a different outcome?

In our first and second examples, since different people are making selections in each step, if we changed the order, different people would end up with different candy bars, so order does matter in the first two situations. In the third example, since there was only one person selecting, all that matters is which two get selected. The order in which they are selected makes no difference.

Factorial Notation: When we solve counting problems in which repetition is **not** allowed, since choices get used up, the number of options available are reduced by one in each step. Because of this, when we apply the Fundamental Counting Principle to solve these problems, we often end up with products that can be written nicely in what is called *factorial notation*.

Any expression of the form $n!$ is called a **factorial** expression. It is a mathematical shorthand for the product or decreasing positive integers.

For example, $3! = 3 \cdot 2 \cdot 1 = 6$, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, and $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. Notice that factorial expressions get large rather quickly. [What is 10!?]

By definition $0! = 1$. We will say more about why this is sensible later.

Definitions:

• A **permutation** is a counting situation where we select r objects from among n possible choices *noting the order* of each choice. For permutation counting, *order matters* and *choices get used up*.

We use $P(n, r)$ or ${}_nP_r$ to denote a permutation of n objects taken r at a time.

To count permutations, we use the formula: $\boxed{P(n,r) = \dfrac{n!}{(n-r)!}}$

Recall that n is the number of possible choices and r is the number of selections actually made.

This formula makes sense if we think about how we would solve a permutation counting problem using a slot diagram. We start with n possible choices. After making our first choice, there are still $n - 1$ options available. At each step, a new choice is made and the options remaining are reduced by 1.

Example: In how many different orders can the balls be sunk in a game of straight pool?

• A **combination** is a counting situation where we select r objects from among n possible choices in an *unordered* manner. For permutation counting, *order does not matter* and *choices get used up*.

We use $C(n, r)$ or ${}_nC_r$ to denote a combination of n objects taken r at a time.

To count combinations, we use the formula: $\boxed{C(n,r) = \dfrac{n!}{r!(n-r)!}}$

Recall that n is the number of possible choices and r is the number of selections actually made.

This formula makes sense if we think about starting by counting the related permutations counting problem. Then, we need to divide by the number of choices made so that we do not "double count" the same choices made in a different order.

Example: Suppose you win a trip to Hawaii in a radio contest. You are allowed to bring 3 friends with you on the trip. How many different ways could you choose 3 from among your 10 best friends to go with you on the trip?