Math 102
Exam 2 - Version 1
2/24/2012

Name:________________________________

1. (3 points each) Compute the value of each of the following:

   (a) $6!$

   $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 1 = 720$

   (b) $\dfrac{999!}{997!}$

   $\dfrac{999!}{997!} = 999 \cdot 998 = 997,002$

   (c) $C(9,5)$

   $C(9,5) = \frac{9!}{5!4!} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 126$

   (d) $P(7,3)$

   $P(7,3) = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 7 \cdot 6 \cdot 5 = 210$

2. (6 points) Suppose you need to choose an identification code consisting of two numbers followed by 2 letters. How many possible codes can be formed if you **can** repeat any number and any letter as many times as you want. (For example: 55BB would be allowed).

   Since we are building a, ID code by selecting two letters and two numbers (in this case with repetition allowed) we will make use of a slot diagram. Each choice will get its own slot, so there will be four slots.

   Recall that there are 26 letters in the alphabet and there are 10 possible digits that can be chosen.

   Since repetition is allowed, we compute $10 \cdot 10 \cdot 26 \cdot 26 = 67,600$.

   Therefore, there are 67,600 possible ID codes consisting of 2 numbers followed by two letters when repetition of both digits and letters is allowed.

3. (6 points) Suppose you need to choose an identification code consisting of two numbers followed by 2 letters. How many possible codes can be formed if you **can** repeat any number but you **cannot** repeat a letter (For example: 33BB and 47CC would both **not** be allowed, but 44QA would be allowed).

   We will make use of a slot diagram. Each choice will get its own slot, so there will be four slots. In this case, there is a restriction on the second letter chosen since repetition of letters is not allowed.

   Since repetition of digits is allowed, but repetition of letters is not, we compute $10 \cdot 10 \cdot 26 \cdot 25 = 65,000$.

   Therefore, there are 65,000 possible ID codes consisting of 2 numbers followed by two letters when repetition of digits is allowed, but repetition of letters is not allowed.

4. (6 points) Suppose you need to choose an identification code consisting of two numbers followed by 2 letters. How many possible codes can be formed if you **cannot** repeat any letters, and the **sum** of the numbers used must add up to 5 (For example: 23AY would be allowed, but 23AA and 57XD would both **not** be allowed.

   We will again make use of a slot diagram. Each choice will get its own slot, but since the condition for choosing the numbers requires the sum to equal 5, it turns out to be better to use three slots: choosing a pair of numbers, then choosing each additional letter.

   Notice that there are six pairs of digits that sum to 5: $(5,0)$, $(4,1)$, $(3,2)$, $(2,3)$, $(1,4)$, and $(0,5)$.

   Then we compute: $6 \cdot 26 \cdot 25 = 3,900$

   Therefore, there are 3,900 possible ID codes consisting of 2 numbers followed by two letters when the sum of the digits is 5, and when repetition of letters is not allowed.

5. Suppose that the Parliament of a certain country consists of 9 monarchists and 6 anarchists.

   (a) (6 points) How many ways can a ruling council of 5 be chosen from among **all** the members of Parliament?

   Notice that we are choosing a ruling council, and there seems to be no distinguished roles for the members of this council. Therefore, we will use combination counting. There are 15 total members of Parliament, and we are choosing 5 of them to be on the council, so we compute:

   $C(15,5) = \frac{15!}{10!5!} = \frac{15\cdot14\cdot13\cdot12\cdot11}{5\cdot4\cdot3\cdot2\cdot1} = \frac{360{,}360}{120} = 3,003$

   Therefore, there are 3,003 distinct ways of choosing a ruling council of 5 from among the 15 members of Parliament.

   (b) (6 points) How many ways can a ruling council of 5 be chosen from among all the members of Parliament if 3 of them must be monarchists and 2 of them must be anarchists?

   Notice that we are still choosing a ruling council with no distinguished roles, so we will use combination counting. Here, we must choose a certain number of members from each party. Then this is a "double combination" counting problem. We compute:

   $C(9,3)\cdot C(6,2) = \frac{9!}{6!3!}\cdot\frac{6!}{4!2!} = \frac{9\cdot8\cdot7}{3\cdot2\cdot1}\cdot\frac{6\cdot5}{2\cdot1} = \frac{504}{6}\cdot\frac{30}{2} = (84)(15) = 1,260$

   Therefore, there are 1,260 distinct ways of choosing a ruling council of 5 consisting of 3 monarchists and 2 anarchists.

6. (6 points each) Suppose that Andrew, Brian, Charles, Doug, Amy, Betty, Carina, and Darlene are all members of the Machiavellian Club on campus. They need to elect their club officers: President, Vice President, and Treasurer.

   (a) How many different ways could these offices be filled?

   Notice that, unlike in the previous problem, each officer selected will be filling a distinct office. Therefore, here we must use either permutation counting, or, equivalently, a slot diagram (both lead to the same solution). Notice that there are 8 members of the Club and no other restrictions are mentioned.

   We will use a slot diagram with 3 slots: one for each office. We compute: $8\cdot7\cdot6 = 336$

   Then there are 336 distinct ways to select club officers from among these 8 club members.

   (b) Suppose that Charles will only agree to serve as an officer if he gets to be President. How many ways could the offices be filled under these circumstances?

   A key observation to make is that, although Charles will refuse to hold any office besides President, that does not necessarily mean that he will end up being selected as President. It only means that if he is not selected as President, then he will refuse the other offices. With this in mind, there are two ways to count this situation.

   First, if we insist on selecting the President first, we must split into two cases: one where Charles is selected as President, and the other when he is not. If Charles is selected as President, then we compute the following: $1\cdot7\cdot6 = 42$. Notice that we already decided who is President, so there was only one choice for that task in this case. Otherwise, Charles will not hold an office, so we are selecting all three officers from the 7 other members: $7\cdot6\cdot5 = 210$. Adding these, there are 252 possible ways to select the officers under these conditions.

   A slightly easier way to count this is to choose the President last. If we count this way, Charles is only an added option in the last slot, so we have: $7\cdot6\cdot6 = 252$, the same answer we found before.

7. Suppose that 4 cards are drawn (at the same time) from a standard deck of 52 cards:

   (a) (6 points) How many different 4 card hands are possible?

   Since all that matters is which cards end up in the hand, we will use combination counting.

   $C(52,4) = \frac{52!}{48!4!} = \frac{52 \cdot 51 \cdot 50 \cdot 49}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{6{,}497{,}400}{24} = 270,725$

   Therefore, there are 270,725 distinct four card hands.

   (b) (6 points) How many 4 card hands contain a three of a kind (but not a four of a kind)?

   This is a slightly more challenging counting problem. As discussed in class, the key is to come up with a clear strategy. We split choosing a hand with a 3-of-a-kind into the following steps. First, choose one of the 13 different types of cards. Second, choose 3 of this card type. Finally, choose a 4th card to fill out the hand, but make sure it is not the 4th card of the previous type so that we do not end up with a 4-of-a-kind (so we have $52 - 4 = 48$ cards to choose from in this last step).

   Computing this, we get: $13 \cdot C(4,3) \cdot 48 = 13 \cdot 4 \cdot 48 = 2,496$.

   Thus, there are 2,496 distinct 4 card hands that contain a 3-of-a-kind (but not a 4-of-a-kind).

8. (a) (5 points) Use roster notation to express the set:
   $A = \{\ x \mid x$ is a letter that occurs in **both** the word *tread* and the word *rent* $\}$

   $A = \{t, r, e\}$

9. Determine whether or not the following sets are well defined:

   (a) (2 points) $J = \{\ j \mid j$ is a joke that was told by Jimmy Fallon on February 1st. $\}$

   This set is well-defined (we can look at a transcript of the things he said on that day – we may not agree on which ones were funny).

   (b) (2 points) $W = \{\ d \mid d$ is a day where the weather is warm. $\}$

   This set is not well defined since "warm" is an ambiguous term.

10. (3 points each) Let $A = \{\ n \mid n$ is a positive odd number less than 12 $\}$

   (a) Write A in roster notation.

   $A = \{1, 3, 5, 7, 9, 11\}$

   (b) Find $n(A)$

   $n(A) = 6$ ($n(A)$ is the cardinal number of the set A – that is, the number of elements in this finite set).

   (c) Give an example of a set B that is equivalent to A but not equal to A.

   There are many possible answers here. I accepted any set with 6 distinct elements.

11. (3 points each) Let $A = \{ x \mid x \text{ is a letter in the word } blend \}$, $B = \{ x \mid x \text{ is a letter in the word } bends \}$, $C = \emptyset$, and $D = \{0, \{0, 1\}\}$ indicate whether the following are **True** or **False** [You DO NOT need to justify your answers.]

(a) $n(A) = n(B)$

TRUE. Notice that $n(A) = 5$ and $n(B) = 5$

(b) $C \subset A$

TRUE. The empty set is a proper subset of any non-empty set.

(c) $0 \in D$

TRUE. Zero is on the list of elements of the set D.

(d) $\{0, 1\} \subset D$

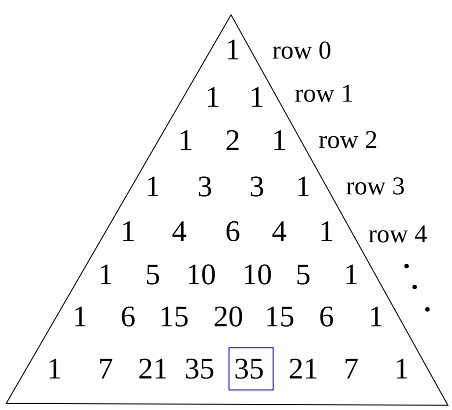
FALSE. Notice that since 1 is not an element of D, the set $\{0, 1\}$ cannot be formed using elements of D. Note that the statement $\{0, 1\} \in D$ is true.

12. (a) (3 points) How many *proper* subsets does the set $\{a, b, c, d, e, f, g\}$ have

Since this set has 7 elements, then it has $2^7 = 128$ subsets. Thus it has $128 - 1 = 127$ proper subsets.

(b) (5 points) Use Pascal's Triangle to find the number of 4 element subsets of $\{a, b, c, d, e, f, g\}$

We begin by constructing Pascal's Triangle down to the 7th row (counting from zero):



We then find the element in this row corresponding to subsets of size 4 (see the marked entry above). From this, we see that this set has 35 distinct subsets of size 4.