Math 327
Basis and Dimension

Recall: A set of vector $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ in a vector space V forms a basis for V if both of the following hold:
(a) $span\, S = V$ (that is, the vector space V is spanned by the set S) (b) S is a linearly independent set.

Examples:

1.

2.

3.

Definition: A vector space V is **finite dimensional** if there is a finite subset of V that is a basis for V. If no such subspace exists, then we say that V is **infinite dimensional**.

Note: If $S_1 = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ is a basis for V, then $S_2 = \{c\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ for $c \neq 0$ is also a basis, so bases aren't unique.

Theorem 4.8: If $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ is a basis for a vector space V then every vector in V can be written in one and only one way as a linear combination of vectors in V.

Proof: Let $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ be a basis for a vector space V and suppose $\vec{v} \in V$. Since S spans V, there is at least one way of expressing $\vec{v}$ as a linear combination in S. Suppose that $\vec{v} = a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_k\vec{v_k}$ and $\vec{v} = b_1\vec{v_1} + b_2\vec{v_2} + \cdots + b_k\vec{v_k}$.

Then $\vec{0} = \vec{v} - \vec{v} = (a_1 - b_1)\vec{v_1} + (a_2 - b_2)\vec{v_2} + \cdots + (a_k - b_k)\vec{v_k}$. Since S is linearly independent, we must have $(a_1 - b_1) = (a_2 - b_2) = \cdots = (a_k - b_k) = 0$. That is, $a_i = b_i$ for all $i = 1 \cdots n$. $\square$.

Theorem 4.9: Let $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ be a set of non-zero vectors in a vector space V and suppose $W = span\, S$. Then some subset of S is a basis for W.

Proof: First, notice that if S is linearly independent, then S is a basis for W.

Suppose that S is linearly dependent. Then, using Theorem 4.7, some vector v_j in S can be written as a linear combination of the other vectors in S. Hence, we may delete this vector from S to obtain a strictly smaller set S_1 that still spans W. If S_1 is linearly independent, then it is a basis for W. Otherwise, we may apply the same procedure to delete a vector from S_1. Since a single non-zero vectors is linearly independent and S is a finite set, repeating this procedure must eventually produce a basis for W.

Note: The proof of Theorem 4.9 suggests the following algorithm for finding a basis of a subspace W of a vector space V:

- **Step 0:** Begin with a set $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ such that $span\, S = W$.
- **Step 1:** Consider the equation $a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_k\vec{v_k} = \vec{0}$. Solve this system for $a_1, a_2, \cdots, a_k$ by representing this system as a matrix and applying the Gauss-Jordan method to put the augmented matrix into reduced row echelon form. If $a_1 = a_2 = \cdots = a_k = 0$, then S is already a basis.
- **Step 2:** If not, find a vector v_j that is a linear combination of the other vectors in S and delete it from the set S, obtaining a smaller set S_1.
- Repeat Steps 1 and 2 for the new set S_1. This process will eventually produce a basis for W.

Special Case: If $V = R^n$ or $V = R_n$, then the following more efficient method can be used:

- **Step 0:** Begin with a set $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ such that $span\, S = W$.
- **Step 1:** Consider the equation $a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_k\vec{v_k} = \vec{0}$. Solve this system for $a_1, a_2, \cdots, a_k$ by representing this system as a matrix and applying the Gauss-Jordan method to put the augmented matrix into reduced row echelon form.
- **Step 2:** The collection of vectors corresponding to the columns that contain a leading 1 form a basis for W.

Theorem 4.10: If $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ is a basis for a vector space V, and $T = \{\vec{w_1}, \vec{w_2}, \cdots, \vec{w_r}\}$ is a linearly independent set in V, then $r \leq n$

Proof:

Corollary 4.1: If $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ and $T = \{\vec{w_1}, \vec{w_2}, \cdots, \vec{w_m}\}$ are bases for a vector space V, then $m = n$.

Proof: Applying Theorem 4.10 to S and T, we have that $m \leq n$. Reversing the roles of the two sets and applying Theorem 4.10 again, we have $n \leq m$. Hence $m = n$. $\square$

Notes: A single vector space V can have many different bases.

Definition: The **dimension** of a non-zero vector space V is the number of vectors in a basis for V. This is well defined by Corollary 4.1. We denote this as: $dim\, V$.

Note: By convention, $dim\, \{\vec{0}\} = 0$.

Examples:

- Both R^n and R_n have dimension n. (What is the dimension of M_{mn}?)
- P_n has dimension $n+1$ (for example, since $P_2 = \{p(t) : p(t) = at^2 + bt + c\}$, then $dim\, P_2 = 3$).
- P is infinite dimensional.

Definition: Let S be a set of vectors in a vector space V. A subset T of S is called a **maximal independent subset** of S if T is a linearly independent set of vectors that is *not* properly contained in any other linearly independent subset of S. Similarly, a **minimal spanning set** of a vector space V is a set S of vectors that spans V and that does not contain any proper subset that spans V.

Corollary 4.2: If $dim\, V = n$, then any maximal independent subset of V contains n vectors.

Corollary 4.3: If a vector space V has dimension n, then any minimal spanning set of V contains n vectors.

Corollary 4.4: If a vector space V has dimension n, then any set of $m > n$ vectors is linearly dependent.

Corollary 4.5: If a vector space V has dimension b, then any set of $m < n$ vectors does not span V.