Math 327
Properties of the Determinant

Theorem 3.1: If A is a matrix, then $det(A) = det\left(A^T\right)$.

Proof: See textbook p. 146

Theorem 3.2: If a matrix B results from matrix A by interchanging two different rows (columns) of A, then $det(B) = -det(A)$.

Proof Sketch:

Interchanging two rows (or columns) of a matrix interchanges the associated elements in each permutation used to compute the determinant of A. We claim that the result of interchanging this pair of elements replaces each odd permutation with an even permutation and vice versa. From this, each permutation that was positive in the original computation of $det(A)$ becomes negative and each permutation that was negative becomes positive. Therefore, $det(B) = -det(A)$.

To see that the same result holds for interchanging columns, notice that if B is derived from A by interchanging two columns, then B^T arises from A^T by interchanging two rows. Then $det(B^T) = -det(A^T)$. However, applying Theorem 3.1, $det(A^T) = det(A)$ and $det(B^T) = det(B)$. Thus $det(B) = -det(A)$.

Theorem 3.3: If two rows (columns) of A are equal, then $det(A) = 0$.

Proof:

Suppose A has two equal rows (columns). Let B be the matrix obtained by interchanging the identical rows (columns). Then $B = A$. However, using Theorem 3.2, $det(B) = -det(A)$. That is, $det(A) = -det(A)$. Therefore, we must have $det(A) = 0$.

Theorem 3.4: If A has a row (column) consisting of all zeros, then $det(A) = 0$.

Proof:

Since an entire row (column) of A has all zeros, each permutation in the computation of $det(A)$ will contain a zero term. Therefore, every product term is zero. Hence $det(A) = 0$.

Theorem 3.5: If B is obtained from A by multiplying a row (column) of A by a real number k, then $det(B) = kdet(A)$.

Proof:

Theorem 3.6: If B is obtained from A by adding to each element of the rth row (column) of A, k times the corresponding element of the sth row (column) of A (with $r \neq s$) then $det(B) = det(A)$.

Proof: See text.

Note: Theorems 3.2, 3.5, and 3.6 tell us exactly what impact carrying our an elementary operation of type I, II, or III on a matrix A does to the value of the determinant of A.

Theorem 3.7: If a matrix A is upper (lower) triangular, then $det(A) = a_{11}a_{22}\cdots a_{nn}$. That is, the determinant is equal to the product of the elements along the main diagonal of A.

Proof Sketch: Notice that, other than the term that comes for choosing the elements along the main diagonal, every other permutation product drawn from A contains at least one zero term.

Note: Taking Theorem 3.7 together with the previous results that describe the impact performing row operations has on the value of the determinant of a matrix suggests that one nice way to compute the determinant of a matrix is to use row (or column) operations to put the matrix into a triangular form, keeping track of how each move changes the determinant, then, compute the value of the determinant of the simplified matrix, and finally, to find the determinant of the original matrix by making the appropriate adjustments. This method is called **computation via reduction to triangular form**.

Example:

Lemma 3.1: If E is an elementary matrix, then $det(EA) = det(E)det(A)$, and $det(AE) = det(A)det(E)$.

Proof Sketch: Let E be an elementary matrix. Notice that $det(I_n) = 1$. Then if E is of type I, then $det(E) = -1$. If E is of type II, the $det(E) = k$, and if E is of type III, then $det(E) = 1$. Combining this with Theorems 3.2, 3.5, and 3.6 above proves the result (many details have been omitted).

Theorem 3.8: If A is an $n \times n$ matrix, then A is nonsingular if and only if $det(A) \neq 0$.

Proof:

First, suppose that A is nonsingular. Then, by Theorem 2.8, $A = E_1 E_2 \cdots E_n$ where each E_i is an elementary matrix. Then, using Lemma 3.1 n times, $det(A) = det(E_1 E_2 \cdots E_n) = det(E_1)det(E_2)\cdots det(E_n) \neq 0$ (as discussed in the sketch of the proof of Lemma 3.1 above, for each i, $det(E_i) \neq 0$). Conversely, suppose A is singular. Then, using Theorem 2.10, A is row equivalent to a matrix B with a row of zeros. Then, by Theorem 3.4, $det(B) = 0$. Therefore, since $A = F_1 F_2 \cdots F_\ell B$, where each F_i is an elementary matrix, $det(A) = det(F_1 F_2 \cdots F_\ell B) = det(F_1)det(F_2)\cdots det(F_\ell)det(B) = 0$.

Corollary 3.1: If A is an $n \times n$ matrix, then $A\vec{x} = \vec{0}$ has a nontrivial solution if and only if $det(A) = 0$.

Theorem 3.9: If A and B are $n \times n$ matrices, then $det(AB) = det(A)det(B)$.

Proof: See text p. 153

Corollary 3.2: If A is nonsingular then $det\left(A^{-1}\right) = \frac{1}{det(A)}$

Proof:
Let A be a nonsingular matrix. Then A^{-1} exists and $AA^{-1} = I_n$. Recall that $det(I_n) = 1$. Then, using Theorem 3.9, $1 = det(I_n) = det(AA^{-1}) = det(A)det(A^{-1})$, or $1 = det(A)det(A^{-1})$. By Theorem 3.8, since A is nonsingular, $det(A) \neq 0$. Hence $det(A^{-1}) = \frac{1}{det(A)}$.