Math 327
The Echelon Form of a Matrix

Definition: An $m \times n$ matrix A is said to be in **reduced row echelon form** if it satisfies the following properties:

- All zero rows, if there are any, appear at the bottom of the matrix.
- The first non-zero entry (from the left) of a non-zero row is a 1 [This entry is called the **leading one** of its row.]
- For each non-zero row, the leading one appears to the right and below and leading ones in preceding rows.
- If a column contains a leading one, then *all other entries in that column are zero.*

Notes:

- A matrix in reduced row echelon form looks like a "staircase" of leading ones descending from the upper left corner of the matrix.
- An $m \times n$ matrix that satisfies the first three properties above is said to be in **row echelon form**.
- By interchanging the roles of rows and columns in the definition given above, we can define the **reduced column echelon form** and the **column echelon form** of an $m \times n$ matrix.

Examples:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -3 & 5 & 7 & 2 \\ 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} C = \begin{bmatrix} 1 & -3 & 5 & 7 & 2 \\ 0 & 0 & 1 & -1 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & -3 & 0 & 7 & 2 \\ 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Definition: An elementary row (column) operation on a matrix A is any one of the following operations:

- **Type I:** Interchange any two rows (columns).
- **Type II:** Multiply a row (column) by a non-zero number.
- **Type III:** Add a multiple of one row (column) to another.

Notation: We will use the following notation to clearly identify the row (column) operations performed on a given matrix.

- **Type I:** $r_i \leftrightarrow r_j$ $(c_i \leftrightarrow c_j)$
- **Type II:** $kr_i \rightarrow r_i$ $(kc_i \rightarrow kc_j)$
- **Type III:** $kr_i + r_j \rightarrow r_j$ $(kc_i + c_j \rightarrow c_j)$

Note: When A is the augmented matrix of a system of linear equations, performing elementary row operations on A is equivalent to performing the operations on linear systems defined in Section 1.1.

Definition: An $m \times n$ matrix B is said to be **row (column) equivalent** to an $m \times n$ matrix A if B can be produced by applying a finite sequence of elementary row (column) operations on A.

Claim: Row equivalence and column equivalence are both equivalence relations on the set of $m \times n$ matrices.

Example:

$$\begin{bmatrix} 1 & 2 & -4 & 7 \\ 2 & -1 & 3 & 5 \\ 3 & 0 & 2 & -6 \end{bmatrix} \quad -2r_1 + r_2 \rightarrow r_2 \quad \begin{bmatrix} 1 & 2 & -4 & 7 \\ 0 & -5 & 11 & -9 \\ 3 & 0 & 2 & -6 \end{bmatrix} \quad -3r_1 + r_3 \rightarrow r_3 \quad \begin{bmatrix} 1 & 2 & -4 & 7 \\ 0 & -5 & 11 & -9 \\ 0 & -6 & 14 & -27 \end{bmatrix}$$

$$-r_3 + r_2 \rightarrow r_2 \quad \begin{bmatrix} 1 & 2 & -4 & 7 \\ 0 & 1 & -3 & 18 \\ 0 & -6 & 14 & -27 \end{bmatrix}$$

Theorem 2.1: Every non-zero matrix $A = [a_{ij}]$ is row (column) equivalent to a matrix in *row echelon form.*

Proof: (Later)

Theorem 2.2: Every non-zero matrix $A = [a_{ij}]$ is row (column) equivalent to a matrix in **reduced** *row echelon form.*
Proof: (Later)

Example:

Recall:

$$\begin{bmatrix} 1 & 2 & -4 & 7 \\ 2 & -1 & 3 & 5 \\ 3 & 0 & 2 & -6 \end{bmatrix} \quad -2r_1 + r_2 \to r_2 \quad \begin{bmatrix} 1 & 2 & -4 & 7 \\ 0 & -5 & 11 & -9 \\ 3 & 0 & 2 & -6 \end{bmatrix} \quad -3r_1 + r_3 \to r_3 \quad \begin{bmatrix} 1 & 2 & -4 & 7 \\ 0 & -5 & 11 & -9 \\ 0 & -6 & 14 & -27 \end{bmatrix}$$

$$-r_3 + r_2 \to r_2 \quad \begin{bmatrix} 1 & 2 & -4 & 7 \\ 0 & 1 & -3 & 18 \\ 0 & -6 & 14 & -27 \end{bmatrix}$$

Continuing this example:

$$\begin{bmatrix} 1 & 2 & -4 & 7 \\ 0 & 1 & -3 & 18 \\ 0 & -6 & 14 & -27 \end{bmatrix} \quad -6r_2 + r_3 \to r_3 \quad \begin{bmatrix} 1 & 2 & -4 & 7 \\ 0 & 1 & -3 & 18 \\ 0 & 0 & -4 & 81 \end{bmatrix} \quad -\tfrac{1}{4}r_3 \to r_3 \quad \begin{bmatrix} 1 & 2 & -4 & 7 \\ 0 & 1 & -3 & 18 \\ 0 & 0 & 1 & -\frac{81}{4} \end{bmatrix}$$

$$3r_3 + r_2 \to r_2 \text{ and } 4r_3 + r_1 \to r_1 \quad \begin{bmatrix} 1 & 2 & 0 & -74 \\ 0 & 1 & 0 & -\frac{171}{4} \\ 0 & 0 & 1 & -\frac{81}{4} \end{bmatrix} \quad -2r_2 + r_1 \to r_1 \quad \begin{bmatrix} 1 & 0 & 0 & \frac{23}{2} \\ 0 & 1 & 0 & -\frac{171}{4} \\ 0 & 0 & 1 & -\frac{81}{4} \end{bmatrix}$$