Math 327
Elementary Matrices and Inverse Matrices

Definition: An $n \times n$ **elementary matrix** of *type I*, *type II*, or *type III* is a matrix obtained from the identity matrix I_n by performing a single elementary row operation (or a single elementary column operation) of type I, II, or III respectively.

Examples:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{7}{4} \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & -11 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Type I ($r_2 \leftrightarrow r_3$) Type II ($-\frac{7}{4}r_3 \to r_3$) Type III ($r_1 - 11r_3 \to r_1$)

Theorem 2.5: Let A be an $m \times n$ matrix. Suppose a single elementary row (column) operation of type I, II, or III is performed on A, yielding the matrix B. Let E be the elementary matrix obtained from I_m (I_n) by performing the same elementary row (column) operation. Then $B = EA$ ($B = AE$).

Proof: Exercise

Examples: Suppose $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and that E_1, E_2, and E_3 are as above.

Then $E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$, $E_2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{7}{4} \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ -\frac{7}{4}g & -\frac{7}{4}h & -\frac{7}{4}i \end{bmatrix}$

$$E_3A = \begin{bmatrix} 1 & 0 & -11 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a - 11g & b - 11h & c - 11i \\ d & e & f \\ g & h & i \end{bmatrix}$$

Theorem 2.6: If A and B are $m \times n$ matrices, then A is row (column) equivalent to B if and only if there are elementary matrices $E_1, e_2, \cdots, E_k$ such that $B = E_kE_{k-1} \cdots E_2E_1A$ ($B = AE_1E_2 \cdots E_{k-1}E_k$)

Proof: (row case)

If A is row equivalent to B, then B is the result of applying a finite sequence of elementary row operations to the matrix A. For $i = 1$ to k, let E_i be the elementary matrix corresponding to the ith elementary row operation in the sequence. Then, using Theorem 2.5 k times, we have $B = E_kE_{k-1} \cdots E_2E_1A$.
Conversely, suppose that $B = E_kE_{k-1} \cdots E_2E_1A$ for some sequence of elementary matrices. Then if we start from A and apply the elementary row operations the correspond to each elementary matrix in order, we will obtain the matrix B. Thus A and B are row equivalent.
Theorem 2.7 An Elementary Matrix E is nonsingular, and E^{-1} is an elementary matrix of the same type.

Proof Sketch:

Type I: We claim that the matrix E corresponding to the elementary row operation $r_i \leftrightarrow r_j$ is its own inverse.

Type II: We claim that the inverse of the matrix E corresponding to the elementary row operation $cr_i \leftrightarrow r_j$ with $c \neq 0$ is the matrix corresponding to the operation $\frac{1}{c}r_i \to r_i$.

Type III: We claim that the inverse of the matrix E corresponding to the elementary row operation $cr_i + r_j \leftrightarrow r_j$ is the matrix corresponding to the operation $-cr_i + r_j \leftrightarrow r_j$.

Lemma 2.1 Let A be an $n \times n$ matrix and let the homogeneous system $A\vec{x} = \vec{0}$ have only the trivial solution $\vec{x} = \vec{0}$. Then A is row equivalent to I_n (that is, the reduced row echelon form of A is I_n.)

Proof:

Theorem 2.8 A is nonsingular if and only if A is the product of elementary matrices.

Proof:

First, suppose that A is a product of the elementary matrices $E_1, E_2, \cdots, E_k$. Then $A = E_1 E_2 \cdots E_{k-1} E_k$. By Theorem 2.7, each E_i is non-singular. By Theorem 1.6, the product of two non-singular matrices is non-singular. Hence A is non-singular.

Next, suppose that A is non-singular. Consider the system $A\vec{x} = \vec{0}$. Then $A^{-1}A\vec{x} = A^{-1}\vec{0} = \vec{0}$, so $I_n\vec{x} = \vec{0}$, or $\vec{x} = \vec{0}$. Thus $A\vec{x} = \vec{0}$ has only the trivial solution. Applying Lemma 2.1, A is row equivalent to I_n. Then there is a sequence of elementary matrices such that $I_n = E_k E_{k-1} \cdots E_2 E_1 A$. Then $A = (E_k E_{k-1} \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$. By Theorem 2.7, the inverse of an elementary matrix is an elementary matrix. Thus A is a product of elementary matrices. $\square$.

Corollary 2.2 A is non-singular if and only if A is row equivalent to I_n.

Proof: See text.

Theorem 2.9 The homogeneous system of n linear equations in n unknowns $A\vec{x} = \vec{0}$ has a non-trivial solution if and only if A is singular.

Summary: The following statements are all equivalent for an $n \times n$ matrix A:

- A is non-singular.
- $A\vec{x} = \vec{0}$ has only the trivial solution.
- A is row (column) equivalent to I_n (i.e. the reduced row echelon form of A is I_n).
- The linear system $A\vec{x} = \vec{b}$ has a unique solution for every $n \times 1$ matrix $\vec{b}$.
- A is the product of elementary matrices.

Why do we care??

During the proof of Theorem 2.8, we showed that if A is nonsingular, then $A = (E_k E_{k-1} \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$.

from this, it follows that $A^{-1} = E_k E_{k-1} \cdots E_2 E_1$. That is, we can find A^{-1} by keeping track of the elementary row operations that are used when putting A into reduced row echelon form. From this, if we start with a partitioned matrix $[A|I_n]$ and preform the row operations to put A into reduced row echelon form, we will end up with a partitioned matrix of the form $[I_n|A^{-1}]$.

Using this method, we can find the inverse of any non-singular $n \times n$ matrix A.

Example: