Math 327
Exam 1 - Practice Problems

1. Find all solutions to each of the following systems of linear equations.

(a) $\begin{cases} x+y=2 \\ 2x-y=10 \end{cases}$

(b) $\begin{cases} x-4y=6 \\ 3x+y=5 \\ 2x+3y=1 \end{cases}$

(c) $\begin{cases} 2x+5y=4 \\ -x+y=5 \\ 3x-y=-10 \end{cases}$

(d) $\begin{cases} x-3y+z=10 \\ 2x+y-z=-3 \\ 5x-8y+2z=27 \end{cases}$

2. Given the homogeneous linear system $\begin{cases} x+2y-z=0 \\ 3x-2y+5z=0 \\ 4x+y-z=0 \end{cases}$
Determine whether or not this system has any nontrivial solutions.

3. Find all values of a for which the following linear system has solutions: $\begin{cases} x+2y+z=a^2 \\ x+y+3z=a \\ 3x+4y+7z=8 \end{cases}$

4. Let $A=\begin{bmatrix} 2 & -1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$, $B=\begin{bmatrix} -1 & 3 \\ 2 & 0 \\ -1 & 4 \end{bmatrix}$, $C=\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 3 \\ -1 & 2 & 0 \end{bmatrix}$, and $D=\begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix}$

If possible, compute the following.

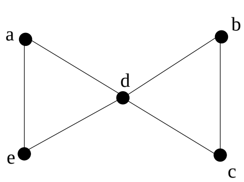
(a) $A+B^T$ (b) $AB+D$ (c) CA^T (d) $CB+A^T$

5. For each of the linear systems in problem 1 above:

(a) Find the coefficient matrix.
(b) Write the linear system in matrix form.
(c) Find the augmented matrix for the system.

6. Rewrite the following as a linear system in matrix form: $x\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}+y\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}+z\begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

7. Find the incidence matrix for the following combinatorial graph.



8. Suppose that $\vec{v}\cdot\vec{w}=0$, with $\vec{v}=\begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$ and $\vec{w}=\begin{bmatrix} x \\ -1 \\ 4 \end{bmatrix}$. Find all possible values for x and y.

9. If possible, find a non-trivial solution to the matrix equation $\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

10. Let $A=[a_{ij}]$ be an $n\times n$ matrix. The **trace** of A, denoted $tr(A)$, is the sum of the entries along the main diagonal of A. That is, $tr(A)=\sum_{i=1}^{n} a_{ii}$. Prove the following:

(a) $Tr(A+B)=Tr(B+A)$ (b) $Tr(A^T)=Tr(A)$. (c) $Tr(A^TA)\geq 0$

11. Prove each of the following.

    (a) Theorem 1.1a
    (b) Theorem 1.2c
    (c) Theorem 1.3b
    (d) Theorem 1.4d

12. Let $A = \begin{bmatrix} -1 & 0 & 4 \\ 3 & -2 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 3 \\ 4 & -1 & 3 \end{bmatrix}$. Show that Theorem 1.4c holds for A and B.

13. Give a nontrivial example of each of the following:

    (a) A diagonal matrix
    (b) An upper triangular matrix
    (c) A symmetric matrix
    (d) A skew symmetric matrix

14. Show that the product of any two diagonal matrices is a diagonal matrix.

15. Show that the sum of any two lower triangular matrices is a lower triangular matrix.

16. Prove or Disprove: For any $n \times n$ matrix A, $A^T A = AA^T$

17. Let A and B be symmetric matrices. Show that AB is symmetric if and only if $AB = BA$.

18. For each matrix A given, either find A^{-1} or show that A is singular.

    (a) $A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$

19. Show that for any $n \times n$ matrix A, $A + A^T$ is symmetric.

20. Let $A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$

    (a) Find A^{-1}
    (b) Use A^{-1} to solve the equation $A\vec{x} = \vec{b}$ if: (i) $\vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (ii) $\vec{b} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$
    (c) Use A^{-1} to solve the equation $A^2\vec{x} = \vec{b}$ if $\vec{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

21. Prove Theorem 1.7

22. Suppose that $f : R^2 \to R^2$ is defined by $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

    (a) Find the image of $\vec{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and then graph both $\vec{u}$ and its image
    (b) Find the image of $\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and then graph both $\vec{v}$ and its image.
    (c) Give a geometrical description of the transformation f given by A above.

23. Let $f : R^2 \to R^3$ be given by $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & 0 \end{bmatrix}$. Determine whether or not $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ or $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ are in the range of f.

24. (a) Find A if $f(\vec{u}) = A\vec{u}$ defines rotation 45° counterclockwise in the plane.
    (b) Find the image of $\vec{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ under f. Then graph both $\vec{v}$ and $f(\vec{v})$.