Math 327

Exam 1 - Practice Problem Solutions

1. Find all solutions to each of the following systems of linear equations.

(a) $\begin{cases} x + y = 2 \\ 2x - y = 10 \end{cases}$

Adding these equations gives $3x = 12$ so $x = 4$
Then $4 + y = 2$, so $y = -2$
Hence the solution is $x = 4, y = -2$.

(b) $\begin{cases} x - 4y = 6 \\ 3x + y = 5 \\ 2x + 3y = 1 \end{cases}$

Adding the first equation to 4 times the second equation $(12x + 4y = 20)$ gives $13x = 26$ or $x = 2$.
Then $2 - 4y = 6$, so $-4y = -4$. Thus $y = -1$. Since there are three equations, we must check this solution in the third equation.
$2(2) + 3(-1) = 4 - 3 = 1$. Since this checks, then the solution is $x = 2, y = -1$.

(c) $\begin{cases} 2x + 5y = 4 \\ -x + y = 5 \\ 3x - y = -10 \end{cases}$

Adding the first equation to twice the second equation $(-2x + 2y = 20)$ gives $7y = 14$ or $y = 2$.
Then $2x + 10 = 4$, so $2x = -6$. Thus $x = -3$. Since there are three equations, we must check this solution in the third equation.
$3(-3) - (2) = -9 - 2 = -11 \neq -10$. Since this does not check, then there is no solution.

(d) $\begin{cases} x - 3y + z = 10 \\ 2x + y - z = -3 \\ 5x - 8y + 2z = 27 \end{cases}$

Adding the first equation to the second equation gives $3x - 2y = 7$. Adding twice the second equation to the third equation gives $9x - 6y = 21$.

Notice that three times the first of these new equations gives $9x - 6y = 21$. Therefore, we see that this line is common to each of the three planes represented by the original three equations. Therefore, this system of equations has infinitely many solutions. Solving for x in our two variable equation gives $3x = 7 + 2y$ or $x = \frac{2}{3}7 + \frac{7}{3}$. Using the original first equation, $z = 10 + 3y - x$, or, substituting, $z = 10 + 3y - \frac{2}{3}y - \frac{7}{3}$. Then $z = \frac{7}{3}y + \frac{23}{3}$.

Hence, the solutions to this system are all points of the form: $\left(\frac{2}{3}t + \frac{7}{3}, t, \frac{7}{3}t + \frac{23}{3}\right)$.

2. Given the homogeneous linear system $\begin{cases} x + 2y - z = 0 \\ 3x - 2y + 5z = 0 \\ 4x + y - z = 0 \end{cases}$
Determine whether or not this system has any nontrivial solutions.

Adding the first equation to the second equation gives $4x + 4z = 0$ or $x + z = 0$, so $z = -x$. Adding the second equation to twice the third equation gives $11x + 3z = 0$. Then, substituting $z = -x$ gives $11x - 3x = 0$, so $8x = 0$. Thus $x = 0$ and $z = 0$. Finally, if $x = 0$ and $z = 0$, then the original first equation becomes $2y = 0$, so $y = 0$. Therefore, this homogeneous system does not have any non-trivial solutions.

3. Find all values of a for which the following linear system has solutions: $\begin{cases} x + 2y + z = a^2 \\ x + y + 3z = a \\ 3x + 4y + 7z = 8 \end{cases}$

We begin by subtracting the first and second equation. This gives $y - 2z = a^2 - a$. Next, we subtract 3 times the first equation from equation 3. This gives $y - 2z = 8 - 3a$. If we subtract these two equations from each other, we get $0 = a^2 - a + 3a - 8$ or $a^2 + 2a - 8 = 0$. This factors to give $(a + 4)(a - 2) = 0$. Therefore, in order for this system to be satisfiable, we must have $a = -4$ or $a = 2$.

4. Let $A = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 2 & 0 \\ -1 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 3 \\ -1 & 2 & 0 \end{bmatrix}$, and $D = \begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix}$

If possible, compute the following.

(a) $A + B^T$

$$A + B^T = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 0 & 5 \end{bmatrix} + \begin{bmatrix} -1 & 2 & -1 \\ 3 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 6 & 0 & 9 \end{bmatrix}$$

(b) $AB + D$

$$AB + D = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 0 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -8 & 22 \\ -8 & 29 \end{bmatrix} + \begin{bmatrix} -4 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -12 & 24 \\ -5 & 30 \end{bmatrix}$$

(c) CA^T

$$CA^T = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 3 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 18 & 23 \\ 15 & 21 \\ -4 & -3 \end{bmatrix}$$

(d) $CB + A^T$

$$CB + A^T = \begin{bmatrix} -1 & 3 \\ 2 & 0 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 3 \\ -1 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -5 & 19 \\ -3 & 18 \\ 5 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 22 \\ -4 & 18 \\ 9 & 2 \end{bmatrix}$$

5. For each of the linear systems in problem 1 above:

(a) Find the coefficient matrix.

$$A_1 = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & -4 \\ 3 & 1 \\ 2 & 3 \end{bmatrix}, A_3 = \begin{bmatrix} 2 & 5 \\ -1 & 1 \\ 3 & -1 \end{bmatrix}, \text{ and } A_4 = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 1 & -1 \\ 5 & -8 & 2 \end{bmatrix}$$

(b) Write the linear system in matrix form.

$$A_1 = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & -4 \\ 3 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 2 & 5 \\ -1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -10 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 1 & -1 \\ 5 & -8 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \\ 27 \end{bmatrix}$$

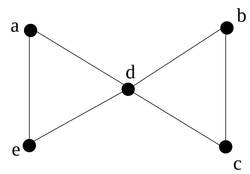
(c) Find the augmented matrix for the system.

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & -1 & 10 \end{array}\right], \left[\begin{array}{cc|c} 1 & -4 & 6 \\ 3 & 1 & 5 \\ 2 & 3 & 1 \end{array}\right], \left[\begin{array}{cc|c} 2 & 5 & 4 \\ -1 & 1 & 5 \\ 3 & -1 & -10 \end{array}\right], \text{ and } \left[\begin{array}{ccc|c} 1 & -3 & 1 & 10 \\ 2 & 1 & -1 & -3 \\ 5 & -8 & 2 & 27 \end{array}\right]$$

6. Rewrite the following as a linear system in matrix form: $x \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + z \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} -1 & 2 & 4 \\ 0 & 1 & 3 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

7. Find the incidence matrix for the following combinatorial graph.



Listing the vertices the graph alphabetically, we obtain the following incidence matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

8. Suppose that $\vec{v} \cdot \vec{w} = 0$, with $\vec{v} = \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} x \\ -1 \\ 4 \end{bmatrix}$. Find all possible values for x and y.

Since $\vec{v} \cdot \vec{w} = 0$, $(1)(x) + (x)(-1) + (y)(4) = x - x + 4y = 4y = 0$. Then $y = 0$. Notice that x can be any real number.

9. If Possible, find a non-trivial solution to the matrix equation $\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Taking the product, we obtain the following system: $\begin{cases} x_1 + 2x_2 = 0 \\ 3x_1 - 4x_2 = 0 \end{cases}$

Twice the first equation is: $2x_1 + 4x_2 = 0$, so adding this to the second equation gives $5x_1 = 0$, so $x_1 = 0$. But then $0 + 2x_2 = 0$, so $x_2 = 0$. Therefore, this homogeneous system has no non-trivial solutions.

10. Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **trace** of A, denoted $tr(A)$, is the sum of the entries along the main diagonal of A. That is, $tr(A) = \sum_{i=1}^{n} a_{ii}$. Prove the following:

(a) $Tr(A + B) = Tr(B + A)$

Let $A = [a_{ij}]$ and $B = [b_{ij}]$. Then $A + B = [a_{ij} + b_{ij}]$ while $B + A = [b_{ij} + a_{ij}]$.

Therefore, applying the definition of trace, $Tr(A + B) = \sum_{i=1}^{n} (a_{ii} + b_{ii}) = \sum_{i=1}^{n} (b_{ii} + a_{ii}) = Tr(B + A)$

(b) $Tr(A^T) = Tr(A)$.

Let $A = [a_{ij}]$. Then $Tr(A) = \sum_{i=1}^{n} (a_{ii})$. Similarly, $A^T = [a_{ji}]$, so $Tr(A^T) = \sum_{i=1}^{n} (a_{ii})$. Hence $Tr(A) = Tr(A^T)$.

(c) $Tr(A^T A) \geq 0$

Let $A = [a_{ij}]$, let $A^T = B = [b_{ij}]$, and let $A^T A = C = [c_{ij}]$. Notice that for each (i, j), $b_{ij} = a_{ji}$.

By definition, $c_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj} = \sum_{k=1}^{n} a_{ki} a_{kj}$. In particular, when $i = j$, $c_{ii} = \sum_{k=1}^{n} a_{ki} a_{ki} = \sum_{k=1}^{n} (a_{ki})^2$.

Therefore, $Tr(A^T A) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} (a_{ki})^2 \right)$

Since each $(a_{ik})^2 \geq 0$, we must have $Tr(A^T A) \geq 0$.

11. Prove each of the following.

(a) **Theorem 1.1a:** Let A and B be $m \times n$ matrices. Then $B + A = A + B$.

Proof:

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices. Let $A + B = C = [c_{ij}]$ and let $B + A = D = [d_{ij}]$. By definition of matrix addition, for each pair (i, j), $c_{ij} = a_{ij} + b_{ij}$ while $d_{ij} = b_{ij} + a_{ij}$. However, since addition of real numbers is commutative, $a_{ij} + b_{ij} = b_{ij} + a_{ij}$. Therefore, $c_{ij} = d_{ij}$. Thus $C = D$. Therefore, $A + B = B + A$. $\Box$.

(b) **Theorem 1.2c:** If A, B, and C are matrices of appropriate sizes, then $C(A + B) = CA + CB$.

Proof:

Let $A = [a_{ij}]$ be an $n \times p$ matrix, $B = [b_{ij}]$ an $n \times p$ matrix, and $C = [c_{ij}]$ an $m \times n$ matrix. Furthermore, let $A + B = D = [d_{ij}]$, let $C(A + B) = CD = E = [e_{ij}]$, let $CA = F = [f_{ij}]$, and let $CB = G = [g_{ij}]$. By definition of matrix addition, for each pair (i, j), $d_{ij} = a_{ij} + b_{ij}$.

Using the definition of matrix multiplication, $f_{ij} = \displaystyle\sum_{k=1}^{n} c_{ik}a_{kj}$ and $g_{ij} = \displaystyle\sum_{k=1}^{n} c_{ik}b_{kj}$. Similarly, $e_{ij} = \sum_{k=1}^{n} c_{ik}d_{kj} = \sum_{k=1}^{n} c_{ik}(a_{kj} + b_{kj}) = \sum_{k=1}^{n} c_{ik}a_{kj} + c_{ik}b_{kj} = \sum_{k=1}^{n} c_{ik}a_{kj} + \sum_{k=1}^{n} c_{ik}b_{kj}$ (using the property 1 of summations). But then $e_{ij} = f_{ij} + g_{ij}$. Therefore, $E = F + G$. That is, $C(A + B) = CA + CB$. $\Box$.

(c) **Theorem 1.3b:** Let r and s be real numbers and A be an $m \times n$ matrix. then $(r + s)A = rA + sA$.

Proof:

Let $A = [a_{ij}]$ and let $(r + s)A = B = [b_{ij}]$. By definition of scalar multiplication, for each pair (i, j), $b_{ij} = (r + s)a_{ij} = ra_{ij} + sa_{ij}$ (using the distributive property of real numbers). Since $rA = [ra_{ij}]$ and $sA = [sa_{ij}]$, it follows that $B = rA + sA$. That is, $(r + s)A = rA + sA$. $\Box$.

(d) **Theorem 1.4d:** Let r be a scalar and A an $m \times n$ matrix. Then $(rA)^T = rA^T$.
Proof:

Let $A = [a_{ij}]$, let $rA^T = B = [b_{ij}]$ and let $(rA)^T = C = [c_{ij}]$. Notice that by definition of matrix transpose and scalar multiplication, for each pair (i, j), $b_{ij} = ra_{ji} = c_{ij}$. it follows that $(rA)^T rA^T$. $\Box$.

12. Let $A = \begin{bmatrix} -1 & 0 & 4 \\ 3 & -2 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 3 \\ 4 & -1 & 3 \end{bmatrix}$. Show that Theorem 1.4c holds for A and B.

Notice that $AB = \begin{bmatrix} -1 & 0 & 4 \\ 3 & -2 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 3 \\ 4 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 15 & -7 & 10 \\ -3 & 7 & -6 \\ 2 & 1 & -1 \end{bmatrix}$

Then $(AB)^T = \begin{bmatrix} 15 & -3 & 2 \\ -7 & 7 & 1 \\ 10 & -6 & -1 \end{bmatrix}$

On the other hand, $A^T = \begin{bmatrix} -1 & 3 & 1 \\ 0 & -2 & -1 \\ 4 & -2 & 0 \end{bmatrix}$ and $B^T = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 3 & 3 \end{bmatrix}$

Hence $B^T A^T = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} -1 & 3 & 1 \\ 0 & -2 & -1 \\ 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 15 & 13 & 2 \\ -7 & 3 & 1 \\ 10 & 6 & -1 \end{bmatrix}$

13. Give a nontrivial example of each of the following:

(a) A diagonal matrix

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) An upper triangular matrix

$$\begin{bmatrix} 5 & -1 & 3 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) A symmetric matrix

$$\begin{bmatrix} 5 & 2 & -1 \\ 2 & -3 & 4 \\ -1 & 4 & 1 \end{bmatrix}$$

(d) A skew symmetric matrix

$$\begin{bmatrix} 5 & 2 & -1 \\ -2 & -3 & 4 \\ 1 & -4 & 1 \end{bmatrix}$$

14. Show that the product of any two diagonal matrices is a diagonal matrix.

Proof:

Let A and B be $n \times n$ diagonal matrices. Suppose that $A = [a_{ij}]$ and $B = [b_{ij}]$. Then, by definition of diagonal, whenever $i \neq j$, we have $a_{ij} = 0$ and $b_{ij} = 0$. Let $AB = C = [c_{ij}]$. By definition of matrix multiplication, $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. Notice that in order for the product $a_{ik}b_{kj} \neq 0$, we must have $i = k = j$. Then, whenever $i \neq j$, $c_{ij} = 0$ and when $i = j$, $c_{ii} = a_{ii}b_{ii}$. Hence $C = AB$ is a diagonal matrix.

15. Show that the sum of any two lower triangular matrices is a lower triangular matrix.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be lower triangular matrices. Let $A + B = C = [c_{ij}]$.
Then for each pair (i, j), $c_{ij} = a_{ij} + b_{ij}$. Now, since A and B are lower triangular, $a_{ij} = 0$ and $b_{ij} = 0$ whenever $i < j$. But then, whenever $i < j$, $a_{ij} + b_{ij} = 0 + 0 = 0$. Hence $A + B$ is also lower triangular.

16. Prove or Disprove: For any $n \times n$ matrix A, $A^T A = AA^T$

This statement is false. To see this, let $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$. Then $A^T = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$.

Therefore, $AA^T = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 10 \end{bmatrix}$ while $A^T A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 10 & -1 \\ -1 & 5 \end{bmatrix}$.

17. Let A and B be symmetric matrices [I forgot to include this necessary hypothesis on the original handout.]. Show that AB is symmetric if and only if $AB = BA$.

Proof:

First, suppose that A and B are symmetric matrices. Consider transpose of the product AB. By Theorem 1.4c, $(AB)^T = B^T A^T$. Then, since A and B are symmetric, $A^T = A$ and $B^T = B$, we have $(AB)^T = B^T A^T = BA$. Given this, if we suppose that AB is symmetric, then $(AB)^T = AB$. But then we have $AB = (AB)^T = B^T A^T = BA$, so $AB = BA$. Conversely, if we suppose that $AB = BA$, then we have that $(AB)^T = B^T A^T = BA = AB$, hence $(AB)^T = AB$. Thus AB is symmetric.

18. For each matrix A given, either find A^{-1} or show that A is singular.

(a) $A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$

Suppose A^{-1} exists. Let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Multiplying this out, we get the equations: $\begin{cases} 3a - c = 1 \\ 2a + 5c = 0 \end{cases}$ and $\begin{cases} 3b - d = 0 \\ 2b + 5d = 1 \end{cases}$

Adding five times the first equations to the second equations gives: $17a = 5$ and $17b = 1$. Then $a = \frac{5}{17}$ and $b = \frac{1}{17}$.

Using these, $\frac{15}{17} - c = 1$, so $c = \frac{15}{17} - 1 = -\frac{2}{17}$, and $\frac{3}{17} - d = 0$, so $d = \frac{3}{17}$.

Hence $A^{-1} = \begin{bmatrix} \frac{5}{17} & \frac{1}{17} \\ -\frac{2}{17} & \frac{3}{17} \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$

Suppose A^{-1} exists. Let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Multiplying this out, we get the equations: $\begin{cases} 2a - c = 1 \\ -4a + 2c = 0 \end{cases}$ and $\begin{cases} 2b - d = 0 \\ -4b + 2d = 1 \end{cases}$

If we add twice the first equation to the second equation in each pair, we get $0 = 2$ and $0 = 1$, which is impossible, so A^{-1} does not exist. Hence A is singular.

19. Show that for any $n \times n$ matrix A, $A + A^T$ is symmetric.

Let $A = [a_{ij}]$, let $A^T = B = [b_{ij}]$, and let $A + A^T = C = [c_{ij}]$. Notice that for each (i, j), $b_{ij} = a_{ji}$.
By definition, for any pair (i, j), $c_{ij} = a_{ij} + b_{ij} = a_{ij} + a_{ji}$. Therefore, $c_{ji} = a_{ji} + b_{ji} = a_{ji} + a_{ij}$. Thus $c_{ij} = c_{ji}$.
Therefore, $\left(A + A^T\right)^T = A + A^T$. $\square$.

A slightly more clever way to do this proof is to note that, using the second property of the transpose operation, $\left(A + A^T\right)^T = A^T + (A^T)^T$, which by the first property of the transpose operation equals $A^T + A$, which by commutativity of addition is just $A + A^T$. Hence $A + A^T$ is symmetric.

20. Let $A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$

(a) Find A^{-1}

Suppose A^{-1} exists. Let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Multiplying this out, we get the equations: $\begin{cases} 2a + c = 1 \\ -3a + 4c = 0 \end{cases}$ and $\begin{cases} 2b + d = 0 \\ -3b + 4d = 1 \end{cases}$

Adding -4 times the first equations to the second equations gives: $-11a = -4$ and $-11b = 1$. Then $a = \frac{4}{11}$ and $b = -\frac{1}{11}$.

Using these, $\frac{8}{11} + c = 1$, so $c = 1 - \frac{8}{11} = \frac{3}{11}$, and $-\frac{2}{17} + d = 0$, so $d = \frac{2}{11}$.

Hence $A^{-1} = \begin{bmatrix} \frac{4}{17} & -\frac{1}{11} \\ \frac{3}{11} & \frac{2}{11} \end{bmatrix}$

(b) Use A^{-1} to solve the equation $A\vec{x} = \vec{b}$ if: (i) $\vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (ii) $\vec{b} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

(i) Recall that if $A\vec{x} = \vec{b}$, then $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} \frac{4}{11} & -\frac{1}{11} \\ \frac{3}{11} & \frac{2}{11} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{11} \\ \frac{8}{11} \end{bmatrix}$.

(ii) Recall that if $A\vec{x} = \vec{b}$, then $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} \frac{4}{11} & -\frac{1}{11} \\ \frac{3}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{16}{11} \\ -\frac{1}{11} \end{bmatrix}$

(c) Use A^{-1} to solve the equation $A^2\vec{x} = \vec{b}$ if $\vec{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

If $A^2\vec{x} = \vec{b}$, then $\vec{x} = A^{-1}A^{-1}\vec{b} = \begin{bmatrix} \frac{4}{17} & -\frac{1}{11} \\ \frac{3}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} \frac{4}{17} & -\frac{1}{11} \\ \frac{3}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{17} & -\frac{1}{11} \\ \frac{3}{11} & \frac{2}{11} \end{bmatrix} \begin{bmatrix} -\frac{6}{11} \\ \frac{1}{11} \end{bmatrix} = \begin{bmatrix} -\frac{25}{121} \\ -\frac{16}{121} \end{bmatrix}$

21. Prove Theorem 1.7

Theorem 1.7: If A is an $n \times n$ nonsingular matrix, then A^{-1} is nonsingular and $\left(A^{-1}\right)^{-1} = A$.

Proof:

Let A be a nonsingular matrix. Then there is an inverse matrix A^{-1} so that $AA^{-1} = I_n$. Now, by definition, in order for A^{-1} to be nonsingular, there must be a matrix B such that $A^{-1}B = BA^{-1} = I_n$. Since A satisfies this property, and since, by Theorem 1.5, the inverse of a matrix is unique whenever it exists, we must have $\left(A^{-1}\right)^{-1} = A$. $\square$.

22. Suppose that $f : R^2 \to R^2$ is defined by $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

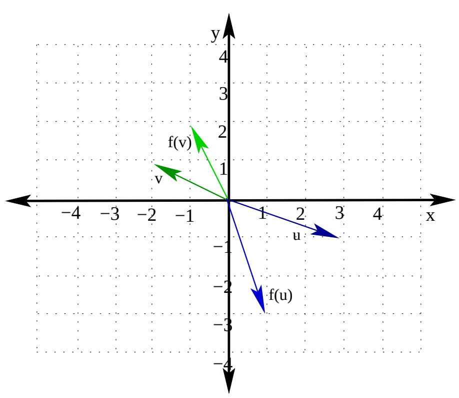
(a) Find the image of $\vec{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and then graph both $\vec{u}$ and its image

$f(\vec{u}) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

See graph below.

(b) Find the image of $\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and then graph both $\vec{v}$ and its image.

$f(\vec{v}) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.



(c) Give a geometrical description of the transformation f given by A above.

In general, $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ -x \end{bmatrix}$. This can be thought of geometrically as first reflecting across the line $y = x$ and then reflecting about the origin.

23. Let $f : R^2 \to R^3$ be given by $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & 0 \end{bmatrix}$. Determine whether or not $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ or $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ are in the range of f.

First, suppose that $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in the range of f. Then for some pair (x, y), we must have $\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

From this, we must have $\begin{cases} x + 2y = 1 \\ -x = 2 \\ x = 3 \end{cases}$ Since we cannot have both $x = -2$ and $x = 3$, this is impossible, so we know that $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is **not** in the range of f.

Next, suppose that $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ is in the range of f. Then for some pair (x, y), we must have $\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

From this, we must have $\begin{cases} x + 2y = 0 \\ -x = 1 \\ x = 2 \end{cases}$ Since we cannot have both $x = -1$ and $x = 2$, this is also impossible, so we know that $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is **not** in the range of f.

24. (a) Find A if $f(\vec{u}) = A\vec{u}$ defines rotation $45°$ counterclockwise in the plane.

Recall that a rotation in the plane is defined by $A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$. Here, we have $\phi = 45°$, so $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$.

(b) Find the image of $\vec{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ under f. Then graph both $\vec{v}$ and $f(\vec{v})$.

$$f(\vec{v}) = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3\sqrt{2} \end{bmatrix}.$$