Math 327
Exam 3 - Practice Problems

1. For each of the following matrices, find the adjoint.

   (a) $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$

   Notice that $A_{11} = 4$, $A_{12} = (-1)3 = -3$, $A_{21} = (-1)(-1) = 1$, and $A_{22} = 2$. Thus $adj A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$

   Notice that $A_{11} = 6$, $A_{12} = (-1)(-4) = 4$, $A_{21} = (-1)(-3) = 3$, and $A_{22} = 2$. Thus $adj A = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$

   (c) $\begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 2 \\ 0 & 1 & -3 \end{bmatrix}$

   Notice that $A_{11} = \begin{vmatrix} 4 & 2 \\ 1 & -3 \end{vmatrix} = -14$, $A_{12} = (-1)\begin{vmatrix} 3 & 2 \\ 0 & -3 \end{vmatrix} = 9$, $A_{13} = \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} = 3$

   $A_{21} = (-1)\begin{vmatrix} -1 & 0 \\ 1 & -3 \end{vmatrix} = -3$, $A_{22} = \begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} = -6$, $A_{23} = (-1)\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = -2$

   $A_{31} = \begin{vmatrix} -1 & 0 \\ 4 & 2 \end{vmatrix} = -2$, $A_{32} = (-1)\begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix} = -4$, $A_{33} = \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = 11$

   Thus $adj A = \begin{bmatrix} -14 & -3 & -2 \\ 9 & -6 & -4 \\ 3 & -2 & 11 \end{bmatrix}$

   (d) $\begin{bmatrix} -1 & 4 & 3 \\ 2 & -1 & 5 \\ 2 & 6 & 16 \end{bmatrix}$

   Notice that $A_{11} = \begin{vmatrix} -1 & 5 \\ 6 & 16 \end{vmatrix} = -46$, $A_{12} = (-1)\begin{vmatrix} 2 & 5 \\ 2 & 16 \end{vmatrix} = -22$, $A_{13} = \begin{vmatrix} 2 & -1 \\ 2 & 6 \end{vmatrix} = 14$

   $A_{21} = (-1)\begin{vmatrix} 4 & 3 \\ 6 & 16 \end{vmatrix} = -46$, $A_{22} = \begin{vmatrix} -1 & 3 \\ 2 & 16 \end{vmatrix} = -22$, $A_{23} = (-1)\begin{vmatrix} -1 & 4 \\ 2 & 6 \end{vmatrix} = 14$

   $A_{31} = \begin{vmatrix} 4 & 3 \\ -1 & 5 \end{vmatrix} = 23$, $A_{32} = (-1)\begin{vmatrix} -1 & 3 \\ 2 & 5 \end{vmatrix} = 11$, $A_{33} = \begin{vmatrix} -1 & 4 \\ 2 & -1 \end{vmatrix} = -7$

   Thus $adj A = \begin{bmatrix} -46 & -46 & 23 \\ -22 & -22 & 11 \\ 14 & 14 & -7 \end{bmatrix}$

2. For each matrix from problem 1, find $A(adj A)$.

   (a) $A(adj A) = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$

   (b) $A(adj A) = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

   (c) $A(adj A) = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 2 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} -14 & -3 & -2 \\ 9 & -6 & -4 \\ 3 & -2 & 11 \end{bmatrix} = \begin{bmatrix} -37 & 0 & 0 \\ 0 & -37 & 0 \\ 0 & 0 & -37 \end{bmatrix}$

(d) $A(adj A) = \begin{bmatrix} -1 & 4 & 3 \\ 2 & -1 & 5 \\ 2 & 6 & 16 \end{bmatrix} \begin{bmatrix} -46 & -46 & 23 \\ -22 & -22 & 11 \\ 14 & 14 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3. For each matrix from problem 1, either find A^{-1} or show that A is singular.

(a) Using Corollary 3.4, $A^{-1} = \frac{1}{det(A)}(adj A) = \frac{1}{11}(adj A) = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{11} & \frac{1}{11} \\ -\frac{3}{11} & \frac{2}{11} \end{bmatrix}$

(b) Since $A(adj A) = O$, then $det(A) = 0$, so A is singular.

(c) Using Corollary 3.4, $A^{-1} = \frac{1}{det(A)}(adj A) = -\frac{1}{37}(adj A) = -\frac{1}{37}\begin{bmatrix} \frac{14}{37} & \frac{3}{37} & \frac{2}{37} \\ -\frac{9}{37} & \frac{6}{37} & \frac{4}{37} \\ -\frac{3}{37} & \frac{2}{37} & -\frac{11}{37} \end{bmatrix}$

(d) Since $A(adj A) = O$, then $det(A) = 0$, so A is singular.

4. (a) Find the adjoint of A if $A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$

Notice that $A_{11} = \begin{vmatrix} b & b^2 \\ c & c^2 \end{vmatrix} = bc^2 - b^2c$, $A_{12} = (-1)\begin{vmatrix} 1 & b^2 \\ 1 & c^2 \end{vmatrix} = b^2 - c^2$, $A_{13} = \begin{vmatrix} 1 & b \\ 1 & c \end{vmatrix} = c - b$

$A_{21} = (-1)\begin{vmatrix} a & a^2 \\ c & c^2 \end{vmatrix} = a^2c - ac^2$, $A_{22} = \begin{vmatrix} 1 & a^1 \\ 1 & c^1 \end{vmatrix} = c^2 - a^2$, $A_{23} = (-1)\begin{vmatrix} 1 & a \\ 1 & c \end{vmatrix} = a - c$

$A_{31} = \begin{vmatrix} a & a^2 \\ b & b^2 \end{vmatrix} = ab^2 - a^2b$, $A_{32} = (-1)\begin{vmatrix} 1 & a^2 \\ 1 & b^2 \end{vmatrix} = a^2 - b^2$, $A_{33} = \begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix} = b - a$

Thus $adj A = \begin{bmatrix} bc^2 - b^2c & a^2c - ac^2 & ab^2 - a^2b \\ b^2 - c^2 & c^2 - a^2 & a^2 - b^2 \\ c - b & a - c & b - a \end{bmatrix}$

(b) Find A^{-1}

Multiplying, $A(adj A) =$

$$\begin{bmatrix} ab^2 - a^2b - cb^2 + ca^2 + c^2b - c^2a & 0 & 0 \\ 0 & ab^2 - a^2b - cb^2 + ca^2 + c^2b - c^2a & 0 \\ 0 & 0 & ab^2 - a^2b - cb^2 + ca^2 + c^2b - c^2a \end{bmatrix}$$

If we let $k = ab^2 - a^2b - cb^2 + ca^2 + c^2b - c^2a$, then $A^{-1} = \frac{1}{k}(adj A) = \frac{1}{k}\begin{bmatrix} bc^2 - b^2c & a^2c - ac^2 & ab^2 - a^2b \\ b^2 - c^2 & c^2 - a^2 & a^2 - b^2 \\ c - b & a - c & b - a \end{bmatrix}$.

5. Prove that if A is singular, then $A(adj A) = 0$

Proof:

Recall that by Theorem 3.8, if A is singular, then $det(A) = 0$. Recall that by Theorem 3.12, $A(adj A) = det(A)I_n$. Then $A(adj A) = 0I_n = 0$.

6. Use Cramer's Rule to solve each of the following linear systems.

(a) $\begin{cases} x + y = 2 \\ 2x - y = 10 \end{cases}$

$det(A) = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1 - 2 = -3.$

$det(A_1) = \begin{vmatrix} 2 & 1 \\ 10 & -1 \end{vmatrix} = -2 - 10 = -12$ and $det(A_2) = \begin{vmatrix} 1 & 2 \\ 2 & 10 \end{vmatrix} = 10 - 4 = 6.$

Then $x_1 = \frac{det(A_1)}{det(A)} = \frac{-12}{-3} = 4$ and $x_2 = \frac{det(A_2)}{det(A)} = \frac{6}{-3} = -2.$

(b) $\begin{cases} x - 4y = 6 \\ 3x + y = 5 \end{cases}$

$det(A) = \begin{vmatrix} 1 & -4 \\ 3 & 1 \end{vmatrix} = 1 + 12 = 13.$

$det(A_1) = \begin{vmatrix} 6 & -4 \\ 5 & 1 \end{vmatrix} = 6 + 20 = 26$ and $det(A_2) = \begin{vmatrix} 1 & 6 \\ 3 & 5 \end{vmatrix} = 5 - 18 = -13.$

Then $x_1 = \frac{det(A_1)}{det(A)} = \frac{26}{13} = 2$ and $x_2 = \frac{det(A_2)}{det(A)} = \frac{-13}{13} = -1.$

(c) $\begin{cases} 3x - 2y + z = -6 \\ 4x - 3y + 3z = 7 \\ 2x + y - z = -9 \end{cases}$

$det(A) = \begin{vmatrix} 3 & -2 & 1 \\ 4 & -3 & 3 \\ 2 & 1 & -1 \end{vmatrix} = (9) + (-12) + (4) - (-6) - (8) - (9) = -10.$

$det(A_1) = \begin{vmatrix} -6 & -2 & 1 \\ 7 & -3 & 3 \\ -9 & 1 & -1 \end{vmatrix} = (-18) + (54) + (7) - (-18) - (14) - (27) = 20.$

$det(A_2) = \begin{vmatrix} 3 & -6 & 1 \\ 4 & 7 & 3 \\ 2 & -9 & -1 \end{vmatrix} = (-21) + (-36) + (-36) - (14) - (24) - (-81) = -50.$

$det(A_3) = \begin{vmatrix} 3 & -2 & -6 \\ 4 & -3 & 7 \\ 2 & 1 & -9 \end{vmatrix} = (81) + (-28) + (-24) - (36) - (21) - (72) = -100.$

Then $x_1 = \frac{det(A_1)}{det(A)} = \frac{20}{-10} = -2$, $x_2 = \frac{det(A_1)}{det(A)} = \frac{-50}{-10} = 5$ and $x_3 = \frac{det(A_2)}{det(A)} = \frac{-100}{-10} = 10.$

7. For each given pair of points P and Q, find the vector $\overrightarrow{PQ}$ and then sketch this vector.

(a) $P(-1, 2)$, $Q(3, -5)$

$\overrightarrow{PQ} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$

(b) $P(-1, 0, 3)$, $Q(1, 2, 4)$

$\overrightarrow{PQ} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

(c) $P(3, -4, 1)$, $Q(-1, 4, 0)$

$\overrightarrow{PQ} = \begin{bmatrix} -4 \\ 8 \\ -1 \end{bmatrix}$

8. Determine the tail of the vector $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$ if:

(a) The head is $(1, 2, 3)$

$P = (-1, 3, -2)$

(b) The head is $(3, -2, 0)$

$P = (1, -1, -5)$

(c) Find the head if the tail is $(1, 2, 3)$

$P = (3, 1, 8)$

9. Let $\vec{u} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$. Find:

(a) $\vec{u} - \vec{v}$

$$\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ -3 \end{bmatrix}$$

(b) $3\vec{u} + 4\vec{v}$

$$\begin{bmatrix} 9 \\ 0 \\ -4 \end{bmatrix} + \begin{bmatrix} -4 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 20 \\ 5 \end{bmatrix}$$

(c) $\vec{w}$ if $\vec{u}+\vec{v}+\vec{w}=\vec{0}$.

$$\begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}, \text{ so } \vec{w} = \begin{bmatrix} -2 \\ -5 \\ -1 \end{bmatrix}.$$

10. If possible, find scalars c_1, c_2 and c_3 not all zero such that $c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} -4 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Considering the associated homogeneous system, we have: $\left[\begin{array}{ccc|c} 1 & 3 & -4 & 0 \\ 2 & -1 & 3 & 0 \\ 3 & 2 & -1 & 0 \end{array}\right] \xrightarrow[r_3 - 3r_1 \to r_3]{r_2 - 2r_1 \to r_2} \left[\begin{array}{ccc|c} 1 & 3 & -4 & 0 \\ 0 & -7 & 11 & 0 \\ 0 & -7 & 11 & 0 \end{array}\right]$

$$\xrightarrow[-\frac{1}{7}r_2 \to r_2]{r_3 - r_2 \to r_3} \left[\begin{array}{ccc|c} 1 & 3 & -4 & 0 \\ 0 & 1 & -\frac{11}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] r_1 - 3r_2 \to r_1 \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{7} & 0 \\ 0 & 1 & -\frac{11}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From this, if we set $c_3 = t$, we must have $c_1 = -\frac{5}{7}t$ and $c_2 = \frac{11}{7}t$. Therefore, if we take $z = 7$, then $c_1 = -5$, $c_2 = 11$, and $c_3 = 7$ is one possible solution.

11. If possible, find scalars c_1, c_2 and c_3 not all zero such that $c_1 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Considering the associated homogeneous system, we have: $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 3 & 2 & -1 & 0 \\ 1 & 5 & -2 & 0 \end{array}\right] r_1 \leftrightarrow r_3 \left[\begin{array}{ccc|c} 1 & 5 & -2 & 0 \\ 3 & 2 & -1 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right] \xrightarrow[r_3 - 2r_1 \to r_3]{r_2 - 3r_1 \to r_2}$

$$\left[\begin{array}{ccc|c} 1 & 5 & -2 & 0 \\ 0 & -13 & 5 & 0 \\ 0 & -11 & 7 & 0 \end{array}\right] -\frac{1}{13}r_2 \to r_2 \left[\begin{array}{ccc|c} 1 & 5 & -2 & 0 \\ 0 & 1 & -\frac{5}{13} & 0 \\ 0 & -11 & 7 & 0 \end{array}\right] r_3 + 11r_2 \to r_3 \left[\begin{array}{ccc|c} 1 & 5 & -2 & 0 \\ 0 & 1 & -\frac{5}{13} & 0 \\ 0 & 0 & \frac{36}{13} & 0 \end{array}\right]$$

From this, if we see that the coefficient matrix is non-singular (its determinant in non-zero), so the trivial solution is the only possible solution.

12. Let V be the set of all functions of the form $f(x) = re^{kx}$, where r, k are real numbers. Let $\oplus$ be defined as $r_1e^{k_1x} \oplus r_2e^{k_2x} = r_1r_2e^{(k_1+k_2)x}$ Let $\odot$ be defined as $c \odot re^{kx} = cre^{kx}$ for any real number c. Determine whether or not V is a vector space. If it is, prove that it satisfies each part of the definition of a vector space. If not, show which properties are not satisfied.

(a) Let $r_1e^{k_1x}$ and $r_2e^{k_2x}$ be in V. Then $r_1e^{k_1x} \oplus r_2e^{k_2x} = r_1r_2e^{(k_1+k_2)x}$. Notice that $r_1r_2 \in \mathbb{R}$ and $(k_1+k_2) \in \mathbb{R}$, so V is closed under addition.

(1) Let $\vec{u} = r_1e^{k_1x}$ and $\vec{v} = r_2e^{k_2x}$ be in V. Then $\vec{u} \oplus \vec{v} = r_1e^{k_1x} \oplus r_2e^{k_2x} = r_1r_2e^{(k_1+k_2)x} = (r_2r_1)e^{(k_2+k_1)x}$ by commutativity of real number multiplication and commutativity of real number addition.
Notice that $\vec{v} \oplus \vec{u} = r_2e^{k_2x} \oplus r_1e^{k_1x} = r_2r_1e^{(k_2+k_1)x}$. Hence $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$.

(2) Let $\vec{u} = r_1e^{k_1x}$, $\vec{v} = r_2e^{k_2x}$, and $\vec{w} = r_3e^{k_3x}$ be in V. Then $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = r_1e^{k_1x} \oplus (r_2e^{k_2x} \oplus r_3e^{k_3x}) = r_1e^{k_1x} \oplus r_2r_3e^{(k_2+k_3)x} = r_1(r_2r_3)e^{(k_1+(k_2+k_3))x}$. Using associativity of real number addition and real number multiplication: $= (r_1r_2)r_3e^{((k_1+k_2)+k_3)x} = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$

(3) We define $\vec{0} = 1e^{0x} = 1e^0 = 1$. Notice that if $\vec{u} = r_1e^{k_1x}$, then $\vec{u} \oplus \vec{0} = r_1e^{k_1x} \oplus 1e^{0x} = r_1(1)e^{(k_1+0)x} = r_1e^{k_1x} = \vec{u} = (1)(r_1)e^{(0+k_1)} = 1e^0 \oplus r_1e^{k_1x} = \vec{0} \oplus \vec{u}$.

(4) Suppose $\vec{u} = r_1e^{k_1x}$. If $r_1 \neq 0$ and $\vec{u} \neq \vec{0}$, then we define $-\vec{u} = \frac{1}{r_1}e^{-k_1x}$. Notice that $\vec{u} \oplus -\vec{u} = r_1e^{k_1x} \oplus \frac{1}{r_1}e^{-k_1x} = r_1\frac{1}{r_1}e^{(k_1-k_1)x} = 1e^0 = \vec{0}$. Similarly, $-\vec{u} \oplus \vec{u} = \frac{1}{r_1}e^{-k_1x} \oplus r_1e^{k_1x} = \frac{1}{r_1}r_1e^{(-k_1+k_1)x} = 1e^0 = \vec{0}$. The negative of $\vec{0}$ is $\vec{0}$. Notice that $1e^0 \oplus 1e^0 = (1)(1)e^{(0+0)x} = 1e^0 = \vec{0}$. However, if $r_1 = 0$, then $\vec{u} = 0$. There is no way to define $-\vec{u}$ in this case, since $0 \oplus r_1e^{k_1x} = 0$ for any vector in V. Therefore, this property fails.

(b) Let $\vec{u} = r_1e^{k_1x}$ and $c \in \mathbb{R}$. Then $c \odot \vec{u} = c \odot r_1e^{k_1x} = cr_1e^{k_1x}$. Since cr_1 is a real number, then V is closed under scalar multiplication.

(5) Let $4 \in \mathbb{R}$ and $\vec{u} = 2e^x$, $\vec{v} = 3e^x$. Then $4 \odot (\vec{u} \oplus \vec{v}) = 4(2)(3)e^{(1+1)x} = 24e^{2x}$.
However, $4 \odot \vec{u} = 8e^x$ and $4 \odot \vec{v} = 12e^x$, so $4 \odot \vec{u} \oplus 4 \odot \vec{v} = 8e^x \oplus 12e^x = 96e^{2x}$, so this property also fails.

(6) Let $\vec{u} = e^x$. Then $(2+3) \odot \vec{u} = (2+3)e^x = 5e^x$. However, $2e^x \oplus 3e^x == 6e^{kx}$. Therefore, this property does not hold.

(7) Let $c, d \in \mathbb{R}$ and let $\vec{u} = re^{kx}$. Then $c \odot (d \odot \vec{u}) = c \odot (dr)e^{kx} = c(dr)e^{kx} = (cd)re^{kx}$, where the final equality uses the associativity of real number multiplication. Thus this property holds.

(8) Consider $1 \odot re^{kx} = 1re^{kx} = re^{kx}$. Then this property holds.

13. Let V be the set of all ordered triples (x, y, z) where x, y and z are real numbers. Let $\oplus$ be defined as $(x_1, y_1, z_1) \oplus (x_2, y_2, z_2) = (x_1 + z_2, y_1 + y_2, z_1 + x_2)$ Let $\odot$ be defined as $c \odot (x, y, z) = (cx, cy, cz)$ for any real number c. Determine whether or not V is a vector space. If it is, prove that it satisfies each part of the definition of a vector space. If not, show which properties are not satisfied.

(a) Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be elements of V. Then $(x_1, y_1, z_1) \oplus (x_2, y_2, z_2) = (x_1 + z_2, y_1 + y_2, z_1 + x_2)$. Since the sum of any pair of real numbers is a real number, $(x_1 + z_2, y_1 + y_2, z_1 + x_2)$ is an ordered triple of real numbers. Thus V is closed under addition.

(1) Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be elements of V. Then $(x_1, y_1, z_1) \oplus (x_2, y_2, z_2) = (x_1 + z_2, y_1 + y_2, z_1 + x_2)$
On the other hand, $(x_2, y_2, z_2) \oplus (x_1, y_1, z_1) = (x_2 + z_1, y_2 + y_1, z_2 + x_1)$
Using a specific counterexample, $(2, 3, 4) \oplus (4, 3, 2) = (2 + 2, 3 + 3, 4 + 4) = (4, 6, 8)$, while $(4, 3, 2) \oplus (2, 3, 4) = (4 + 4, 3 + 3, 2 + 2) = (8, 6, 4)$. Therefore, $\oplus$ is not commutative.

(2) Consider $(1, 2, 3)$ and $(4, 5, 6)$ and $(7, 8, 9)$
$(1, 2, 3) \oplus [(4, 5, 6) \oplus (3, 1, 2)] = (1, 2, 3) \oplus (4 + 2, 5 + 1, 6 + 3) = (1, 2, 3) \oplus (6, 6, 9) = (1 + 9, 2 + 6, 3 + 6) = (10, 8, 9)$
while $[(1, 2, 3) \oplus (4, 5, 6)] \oplus (3, 1, 2) = (1+6, 2+5, 3+4) \oplus (3, 1, 2) = (7, 7, 7) \oplus (3, 1, 2) = (7+2, 7+1, 7+3) = (9, 8, 10)$
Therefore, addition is not associative.

(3) Let $\vec{0} = (0, 0, 0)$. Then $(x, y, z) \oplus (0, 0, 0) = (x+0, y+0, z+0) = (x, y, z)$ and $(0, 0, 0) \oplus (x, y, z) = (0+x, 0+y, 0+z) = (x, y, z)$.
Therefore, this vector space does have a zero.

(4) Let (x, y, z) be a vector, and consider $(x, y, z) \oplus (-z, -y, -x)$. Then $(x, y, z) \oplus (-z, -y, -x) = (x - x, y - y, z - z) = (0, 0, 0) = \vec{0}$.
On the other hand, $(-z, -y, -x) \oplus (x, y, z) = (-z + z, +y, -x + x) = (0, 0, 0) = \vec{0}$. Therefore, every element has a negative.

(b) Let (x, y, z) be a vector and $r \in \mathbb{R}$. Then $r \odot (x, y, z) = (rx, ry, rz)$ is still an ordered triple of real numbers, so this vector space is closed under scalar multiplication.

(5) Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be elements of V and $r \in \mathbb{R}$. Then $r \odot [(x_1, y_1, z_1) \oplus (x_2, y_2, z_2)] = r \odot (x_1 + z_2, y_1 + y_2, z_1 + x_2) = (r(x_1 + z_2), r(y_1 + y_2), r(z_1 + x_2)) = (rx_1 + rz_2, ry_1 + ry_2, rz_1 + rx_2)$, where the last equality is due to the distributive property of real numbers.
On the other hand, $r \odot (x_1, y_1, z_1) \oplus r \odot (x_2, y_2, z_2)] = (rx_1, ry_1, rz_1) \oplus (rx_2, ry_2, rz_2) = (rx_1 + rz_2, ry_1 + ry_2, rz_1 + rx_2)$
Therefore, this distributive property holds.

(6) Consider $2, 3 \in \mathbb{R}$ and $(3, 4, 5) \in V$. Then $(2 + 3) \odot (3, 4, 5) = 5 \odot (3, 4, 5) = (15, 20, 25)$.
On the other hand, $2 \odot (3, 4, 5) \oplus 3 \odot (3, 4, 5) = (6, 8, 10) \oplus (9, 12, 15) = (6 + 15, 8 + 12, 10 + 9) = (21, 20, 19)$
Then this property is not satisfied.

(7) Consider $c, d \in \mathbb{R}$ and $(x, y, z) \in V$. Then $(cd) \odot (x, y, z) = ((cd)x, (cd)y, (cd)z)$.
On the other hand, $c \odot (d \odot (x, y, z)) = c \odot (dx, dy, dz) = (c(dx), c(dy), c(dz)) = (cd(x), cd(y), cd(z))$, where the last equality uses the associativity of real number multiplication.
Then this property is satisfied.

(8) Notice that Consider $(1) \odot (x, y, z) = ((1x, 1y, 1z) = (x, y, z)$, so this property is satisfied.

14. Let V be the set of real numbers. Let $\oplus$ be defined as $r \oplus s = rs$. Let $\odot$ be defined as $c \odot r = c + r$ for any real number c. Determine whether or not V is a vector space. If it is, prove that it satisfies each part of the definition of a vector space. If not, show which properties are not satisfied.

(a) Let $r, s \in \mathbb{R}$. Then $r \oplus s = rs$, which, by definition of real number multiplication, is a real number. Then V is closed under $\oplus$.

(1) Let $r, s \in \mathbb{R}$. Then $r \oplus s = rs = sr = s \oplus r$ where the middle equality uses the commutativity of real number multiplication.

(2) Let $r, s, t \in \mathbb{R}$. Then $r \oplus (s \oplus t) = r \oplus (st) = r(st) = rs(t) = (r \oplus s) \oplus t$. We use the associativity of real number multiplication.

(3) Let $r \in \mathbb{R}$ and define $\vec{0} = 1$. Then $r \oplus \vec{0} = r(1) = r = 1(r) = \vec{0} \oplus r$. Hence this property holds.

(4) This property does not hold. If we take $r = 0$, then $rs = 0s = 0$ for any $s \in \mathbb{R}$, but $\vec{0} = 1$. Hence $r = 0$ has no negative.

(b) Let $r \in V$ and $c \in \mathbb{R}$. Then $c \odot r = c + r$. Since the real numbers are closed under addition, this vector space is closed under $\odot$.

(5) Let $c = 2$, $\vec{u} = 3$ and $\vec{v} = 4$. Then $c \odot (\vec{u} \oplus \vec{v}) = 2 \odot (3)(4) = 2 + 12 = 14$.

On the other hand, $c \odot \vec{u} \oplus c \odot \vec{v} = (2+3) \oplus (2+4) = 5 \oplus 6 = (5)(6) = 30$.

Hence this property does not hold.

(6) Let $c = 2$, $d = 3$ and $\vec{u} = 5$. Then $(c + d) \odot \vec{u} = 5 \odot 5 = 5 + 5 = 10$.

On the other hand, $c \odot \vec{u} \oplus d \odot \vec{u} = (2+5) \oplus (3+5) = 7 \oplus 8 = (7)(8) = 56$.

Hence this property does not hold.

(7) Let $c = 3$, $d = 4$, and $\vec{u} = 5$. Then $c \odot (d \odot \vec{u}) = c \odot (4+5) = 3 + (4+5) = 3 + (9) = 12$.

However, $(cd) \odot \vec{u} = 12 \odot 5 = 12 + 5 = 17$. Thus this property does not hold.

(8) Let $r = 5$. Then $1 \cdot 5 = 1 + 5 = 6$. Hence this property does not hold.

15. Prove that a vector space has only one zero vector (that is, the zero of a vector space is unique).

Proof: Let V be a vector space and suppose that $\vec{0}$ and $\hat{0}$ are both zero vectors.

By definition of a zero vector, $\vec{u} \oplus \vec{0} = \vec{0} \oplus \vec{u} = \vec{u}$ for any $\vec{u} \in V$. Similarly, $\vec{u} \oplus \hat{0} = \hat{0} \oplus \vec{u} = \vec{u}$ for any $\vec{u} \in V$. In particular, $\hat{0} \oplus \vec{0} = \hat{0}$ since $\vec{0}$ is a zero vector. But $\hat{0} \oplus \vec{0} = \vec{0}$ since $\hat{0}$ is a zero vector. Thus $\hat{0} = \vec{0}$. Hence the zero vector is unique.

16. Prove that in a vector space, $-1 \odot \vec{u} = -\vec{u}$ for any vector $\vec{u} \in V$.

Proof: Let V be a vector space and let $\vec{u} \in V$. Consider $-1 \odot \vec{u}$.

Notice that $-1 \odot \vec{u} \oplus \vec{u} = -1 \odot \vec{u} \oplus 1 \odot \vec{u}$ by Property (8) of a vector space.

$= (-1 + 1) \odot \vec{u}$ by Property (6) of a vector space.

$= 0 \odot \vec{u}$ by real number addition.

$= \vec{0}$ by part (a) of Theorem 4.2.

Similarly, $\vec{u} \oplus -1 \odot \vec{u} = 1 \odot \vec{u} \oplus -1 \odot \vec{u}$ by Property (8) of a vector space.

$= (1 + (-1)) \odot \vec{u}$ by Property (6) of a vector space.

$= 0 \odot \vec{u}$ by real number addition.

$= \vec{u}$ by part (a) of Theorem 4.2.

Hence $-\vec{u} = -1 \odot \vec{u}$

17. Prove that in any vector space, for a given vector $\vec{u}$, $-(-\vec{u}) = \vec{u}$.

Proof: Let V be a vector space and let $\vec{u} \in V$. Consider $-(-\vec{u})$.

Using the result of problem 16 above, $-\vec{u} = -1 \odot \vec{u}$. Then $-(-\vec{u}) = -(-1 \odot \vec{u}) = -1 \odot (-1 \odot \vec{u})$.

Applying Property (7) of a vector space, $-1 \odot (-1 \odot \vec{u}) = ((-1)(-1)) \odot \vec{u} = 1 \odot \vec{u}$.

Then, applying Property (8) of a vector space, $1 \odot \vec{u} = \vec{u}$. Thus $-(-\vec{u}) = \vec{u}$.

18. Let $V = R^3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where x , y and z are real numbers, and $\oplus$ and $\odot$ are the usual operations. Determine which of the following are subspaces of V:

(a) $W_1 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x - y = z \right\}$

Notice that since $2x - y = z$, if $\vec{u}, \vec{v} \in W_1$, then they have the form: $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ 2x_1 - y_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ 2x_2 - y_2 \end{bmatrix}$

Then $\vec{u} + \vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ 2x_1 - y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ 2x_2 - y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 2x_1 - y_1 + 2x_2 - y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 2(x_1 + x_2) - (y_1 + y_2) \end{bmatrix}$. Thus $\vec{u} + \vec{v} \in W_1$.

Similarly, for any $r \in \mathbb{R}$, $r\vec{u} = r \begin{bmatrix} x_1 \\ y_1 \\ 2x_1 - y_1 \end{bmatrix} = \begin{bmatrix} rx_1 \\ ry_1 \\ r(2x_1 - y_1) \end{bmatrix} = \begin{bmatrix} rx_1 \\ ry_1 \\ 2rx_1 - ry_1 \end{bmatrix}$. Thus $r\vec{u} \in W_1$.

Hence W_1 is a subspace of $V = R^3$.

(b) $W_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y - z = 0 \right\}$

Notice that since $x + y - z = 0$, then $x + y = z$, so if $\vec{u}, \vec{v} \in W_1$, then they have the form: $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ x_1 + y_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ x_2 + y_2 \end{bmatrix}$

Then $\vec{u}+\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ x_1 + y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ x_1 + y_1 + x_2 + y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ (x_1 + x_2) + (y_1 + y_2) \end{bmatrix}$. Thus $\vec{u}+\vec{v} \in W_2$.

Similarly, for any $r \in \mathbb{R}$, $r\vec{u} = r \begin{bmatrix} x_1 \\ y_1 \\ x_1 + y_1 \end{bmatrix} = \begin{bmatrix} rx_1 \\ ry_1 \\ r(x_1 + y_1) \end{bmatrix} = \begin{bmatrix} rx_1 \\ ry_1 \\ rx_1 + ry_1 \end{bmatrix}$. Thus $r\vec{u} \in W_2$.

Hence W_2 is a subspace of $V = R^3$.

(c) $W_3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y = 1 \right\}$

19. Let $V = M_{33}$. Determine which of the following are subspaces of V.

Notice that if $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, then $\vec{u}, \vec{v} \in W_3$. However, $\vec{u}+\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, but $1+1 = 2 \neq 1$, so the condition for being in W_3 is not satisfied.

Since W_3 is not closed under addition, it is not a subspace of R^3.

(a) W_1 is the set of all 3×3 scalar matrices.

Recall that a scalar 3×3 matrix is any matrix of the form: $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ for some $k \in \mathbb{R}$.

Consider the sum of two scalar matrices: $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} + \begin{bmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix} = \begin{bmatrix} a+b & 0 & 0 \\ 0 & a+b & 0 \\ 0 & 0 & a+b \end{bmatrix}$. Since $a+b$ is a constant in $\mathbb{R}$, then W_1 is closed under addition.

Next, consider a scalar multiple of a scalar matrix: $r \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} = \begin{bmatrix} ra & 0 & 0 \\ 0 & ra & 0 \\ 0 & 0 & ra \end{bmatrix}$. Since ra is a constant in $\mathbb{R}$, then W_1 is also closed under scalar multiplication.

Hence W_1 is a subspace of M_{33}

(b) W_2 is the set of all non-singular 3×3 matrices.

Recall that the zero matrix O is singular (it has determinant zero). Let A be a non-singular 3×3 matrix. Since $0 \odot A = O$, W_2 is not closed under scalar multiplication, and hence is not a subspace of M_{33}.

(c) W_3 is the set of all symmetric 3×3 matrices.

Recall that a symmetric 3×3 matrix is any matrix of the form: $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$.

Consider the sum of two symmetric matrices: $\begin{bmatrix} a_1 & b_1 & c_1 \\ b_1 & d_1 & e_1 \\ c_1 & e_1 & f_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 & c_2 \\ b_2 & d_2 & e_2 \\ c_2 & e_2 & f_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 \\ b_1+b_2 & d_1+d_2 & e_1+e_2 \\ c_1+c_2 & e_1+e_2 & f_1+f_2 \end{bmatrix}$.
Notice that the resulting matrix is symmetric. Hence W_3 is closed under addition.

Next, consider a scalar multiple of a symmetric matrix: $r \begin{bmatrix} a_1 & b_1 & c_1 \\ b_1 & d_1 & e_1 \\ c_1 & e_1 & f_1 \end{bmatrix} = \begin{bmatrix} ra_1 & rb_1 & rc_1 \\ rb_1 & rd_1 & re_1 \\ rc_1 & re_1 & rf_1 \end{bmatrix}$. Notice that the resulting matrix is symmetric. Hence W_3 is also closed under scalar multiplication.

Hence W_3 is a subspace of M_{33}

20. Let V be $C(-\infty, \infty)$ with the usual operations. Determine which of the following are subspaces of V.

(a) W_1 is the set of all continuous functions such that $f(0) = 0$.

Let f and g be continuous functions satisfying $f(0) = 0$ and $g(0) = 0$. Consider $(f+g)(t)$. As proven in Calculus I, the sum of two continuous functions is continuous. Moreover, $(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$. Hence W_1 is closed under addition.

Next, let $c \in \mathbb{R}$ and consider $cf(t)$. As proven in Calculus I, a scalar multiple of a continuous function is continuous. Moreover, $cf(0) = c(0) = 0$. Hence W_1 is closed under scalar multiplication. Thus W_1 is a subspace of $C(-\infty, \infty)$.

(b) W_2 is the set of all continuous function such that $f(0) = 1$.

Let $f(t) = t^2 + 1$. Then $f(0) = 0^2 + 1 = 1$. Let $g(t) = 1$. Then $g(0) = 1$. However, $(f+g)(0) = 1 + 1 = 2$. Hence $f + g$ is not in W_2. Since W_2 is not closed under addition, W_2 is not a subspace of $C(-\infty, \infty)$.

(c) W_3 is the set of all differentiable functions.

First and foremost, recall that, as proven in Calculus I, if a function $f(t)$ is differentiable, then $f(t)$ is also continuous, so $W_3 \subseteq C(-\infty, \infty)$.

Let $f(t)$ and $g(t)$ be differentiable functions. As proven in Calculus I, $\frac{d}{dt}(f(t)+g(t)) = f'(t)+g'(t)$. Therefore, the sum of two differentiable functions is differentiable. Similarly, $\frac{d}{dt}(cf(t)) = cf'(t)$, so a scalar multiple of a differentiable function is differentiable. Since the set of differentiable functions is closed under both addition and scalar multiplication, then W_3 is a subspace of $C(-\infty,\infty)$.

(d) W_4 is the set of all constant functions.

Let $f(t) = a$ and $g(t) = b$ be constant functions. Recall that constant functions are continuous on $\mathbb{R}$ (they are differentiable – alternatively, they are zero degree polynomials). Notice that $(f+g)(t) = a+b$, which is still a constant function. Also, if $r \in \mathbb{R}$, then $rf(t) = ra$ which is a constant function. Thus W_4 is closed under both addition and scalar multiplication. Hence W_4 is a subspace of $C(-\infty,\infty)$.

21. Let $v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$. Which of the following vectors are linear combinations of v_1 and v_2?

Let $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ be a vector and suppose that $\vec{v}$ is a linear combination of $\vec{v_1}$ and $\vec{v_2}$. We consider the following matrix related to the linear system $\vec{v} = c_1\vec{v_1} + \vec{v_2}$:

$$\left[\begin{array}{cc|c} 1 & -2 & a \\ -2 & 4 & b \end{array}\right] r_2 + 2r_1 \to r_2 \left[\begin{array}{cc|c} 1 & -2 & a \\ 0 & 0 & 2a+b \end{array}\right]$$

This system has a solution if and only if $2a+b=0$.

(a) $\begin{bmatrix} 6 \\ -12 \end{bmatrix}$

Since $2(6)+(-12)=0$, this vector is a linear combination of $\vec{v_1}$ and $\vec{v_2}$. In fact, $\vec{v} = 6\vec{v_1}$.

(b) $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$

Since $2(3)+(-5) = 1 \neq 0$, this vector is **not** a linear combination of $\vec{v_1}$ and $\vec{v_2}$.

(c) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Since $2(0)+(0)=0$, this vector is a linear combination of $\vec{v_1}$ and $\vec{v_2}$. In fact, $\vec{v} = 0\vec{v_1}$ (we are being a bit silly here – $\vec{0}$ is a linear combination of any set of vectors since we can take the scalars to be all zeros)

22. Let $v_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix}$. Which of the following vectors are linear combinations of v_1, v_2 and v_3?

Let $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be an arbitrary vector in R^3 and consider the related system of equations:

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & a \\ 3 & 2 & -1 & b \\ 1 & 3 & -4 & c \end{array}\right] r_1 \leftrightarrow r_3 \left[\begin{array}{ccc|c} 1 & 3 & -4 & c \\ 3 & 2 & -1 & b \\ 2 & -1 & 3 & a \end{array}\right]$$

$$\overset{r_2 - 3r_1 \to r_2}{r_3 - 2r_1 \to r_3} \left[\begin{array}{ccc|c} 1 & 3 & -4 & c \\ 0 & -7 & 11 & b-3c \\ 0 & -7 & 11 & a-2c \end{array}\right] r_2 - r_3 \to r_3 \left[\begin{array}{ccc|c} 1 & 3 & -4 & c \\ 0 & -7 & 11 & b-3c \\ 0 & 0 & 0 & b-a-c \end{array}\right].$$

From this, we see that this system has a solution if and only if $b-a-c=0$. That is, if $b=a+c$.

(a) $\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$

Using the condition proved above, we have $a = 3$, $b = 6$, and $c = 3$, so $a + c = 3 + 3 = 6 = b$. Hence this vector is a linear combination of $\vec{v_1}$ and $\vec{v_2}$.

(b) $\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$

Using the condition proved above, we have $a = 3$, $b = 4$, and $c = 2$, so $a + c = 3 + 2 = 4 \neq b$. Hence this vector is **not** a linear combination of $\vec{v_1}$ and $\vec{v_2}$.

(c) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Using the condition proved above, we have $a = 0$, $b = 0$, and $c = 0$, so $a + c = 0 + 0 = 0 = b$. Hence this vector is a linear combination of $\vec{v_1}$ and $\vec{v_2}$.

23. If possible, find a non-zero vector in the null space of each of the following vectors:

(a) $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

Consider the matrix associated with the homogeneous system $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 2 & 4 & 0 \end{array}\right] r_1 - r_2 \to r_1 \left[\begin{array}{cc|c} 1 & -5 & 0 \\ 2 & 4 & 0 \end{array}\right] r_2 - 2r_1 \to r_2 \left[\begin{array}{cc|c} 1 & -5 & 0 \\ 0 & 14 & 0 \end{array}\right] \tfrac{1}{14} r_2 \to r_2 \left[\begin{array}{cc|c} 1 & -5 & 0 \\ 0 & 1 & 0 \end{array}\right] r_1 + 5r_2 \to r_1 \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

Then this system has only the trivial solution. That is, $\vec{0}$ is the only vector in the null space of $\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$

Consider the matrix associated with the homogeneous system $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right] r_2 + 2r_1 \to r_2 \left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] -r_1 \to r_1 \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Then this system has solutions of the form: $a - 2b = 0$. For example, if $b = 1$, then $a = 2$. Notice that $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is in the null space of $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$.

(c) $\begin{bmatrix} -3 & 0 & 4 \\ 2 & -1 & 0 \\ 5 & 0 & -2 \end{bmatrix}$

Consider the matrix associated with the homogeneous system $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} -3 & 0 & 4 & 0 \\ 2 & -1 & 0 & 0 \\ 5 & 0 & -2 & 0 \end{array}\right] -r_1 - r_2 \to r_1 \left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 2 & -1 & 0 & 0 \\ 5 & 0 & -2 & 0 \end{array}\right] \begin{array}{c} r_2 - 2r_1 \to r_2 \\ r_3 - 5r_1 \to r_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & -3 & 8 & 0 \\ 0 & 5 & 18 & 0 \end{array}\right]$$

$$\begin{array}{c} -\frac{1}{3} r_2 \to r_2 \\ r_3 - 5r_2 \to r_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & 1 & -\frac{8}{3} & 0 \\ 0 & 0 & \frac{14}{3} & 0 \end{array}\right].$$

Notice that we can see that the determinant of the coefficient matrix is non-zero, so this system has only the trivial solution.

24. Describe the set of **all** vectors in the null space of the matrix $A = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 0 & -1 \\ 1 & 1 & 3 \end{bmatrix}$

Considering the associated homogeneous system, we have: $\left[\begin{array}{ccc|c} -2 & 1 & 4 & 0 \\ 3 & 0 & -1 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right] \; r_2 + r_1 \to r_1 \; \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 3 & 0 & -1 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right] \; \overset{r_2 - 3r_1 \to r_2}{r_3 - r_1 \to r_3}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -3 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] - \frac{1}{3}r_2 \to r_2 \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & \frac{10}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] r_1 - r_2 \to r_1 \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{10}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From this, if we set $z = t$, we must have $x = \frac{1}{3}t$ and $y = -\frac{10}{3}t$. Thus the null space of this matrix is all vectors of the form: $\left[\begin{array}{c} \frac{1}{3}t \\ -\frac{10}{3}t \\ t \end{array}\right]$ for any $t \in \mathbb{R}$.