Math 327
Exam 4 - Practice Problems

1. Let $V = P_3$ and let $S = \{t^3 - 1, t^2 + 1, t + 1\}$. Determine whether or not each of the following are in the span of S:

   (a) $3t^3 + 4t^2 - 2t - 1$

   (b) $5t^3 + 3t^2 - 2t - 3$

2. Let $V = M_{22}$ and let $S = \left\{ \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$. Determine whether or not each of the following are in the span of S:

   (a) $\begin{bmatrix} 7 & -1 \\ 4 & 5 \end{bmatrix}$

   (b) $\begin{bmatrix} 7 & 1 \\ -1 & 7 \end{bmatrix}$

3. Which of the following sets span R_3?

   (a) $S = \{ \begin{bmatrix} -1 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -2 \end{bmatrix} \}$
   (b) $S = \{ \begin{bmatrix} 1 & -2 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 5 & -10 \end{bmatrix} \}$
   (c) $S = \{ \begin{bmatrix} 2 & -3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 3 \end{bmatrix}, \begin{bmatrix} 2 & -4 & -5 \end{bmatrix}, \begin{bmatrix} 3 & -4 & 3 \end{bmatrix} \}$

4. Describe the subspace of R^3 spanned by the set $S = \{ \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \}$

5. For each of the following sets of vectors, determine whether or not the set is linearly independent. For those that are dependent, write one of the vectors as a linear combination of the others.

   (a) $S = \{ \begin{bmatrix} 2 & -3 \end{bmatrix}, \begin{bmatrix} -1 & 3 \end{bmatrix}, \begin{bmatrix} 2 & -4 \end{bmatrix} \}$
   (b) $S = \{t^2 - t + 1, 2t^2 + 5, 2t + 3\}$
   (c) $S = \{t^2 - t, 2t^2 + 5, 2t + 3\}$

6. For which values of c are the vectors $t^2 + 2t + 1$, $2 - t$, and $t^2 + t + c^2$ linearly independent?

7. Suppose that $S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly independent. Let $T = \{\vec{w_1}, \vec{w_2}, \vec{w_3}\}$, where $\vec{w_1} = \vec{v_1} - \vec{v_2}$, $\vec{w_2} = \vec{v_1} + \vec{v_2} + \vec{v_3}$, and $\vec{w_3} = 2\vec{v_1} + \vec{v_3}$. Determine whether or not T is linearly independent.

8. Prove the following: If $S = \{\vec{v_1}, \vec{v_2}\}$ is a linearly independent set and $\vec{v_3}$ is not in the span of S, then $T = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly independent.

9. Let S_1 and S_2 be finite subsets of a vector space V. Suppose that $S_1 \subset S_2$ and that S_1 is linearly independent. Must S_2 be linearly independent?

10. Determine which of the following sets form a basis for R_3. For those that are not, state which part(s) of the definition of a basis are not satisfied.

    (a) $S_1 = \{ \begin{bmatrix} 4 & -1 & 5 \end{bmatrix}, \begin{bmatrix} -2 & 3 & 1 \end{bmatrix} \}$
    (b) $S_2 = \{ \begin{bmatrix} 4 & 3 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 6 \end{bmatrix} \}$
    (c) $S_3 = \{ \begin{bmatrix} -1 & 5 & 3 \end{bmatrix}, \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 & -2 \end{bmatrix} \}$
    (d) $S_4 = \{ \begin{bmatrix} 5 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 0 & -2 \end{bmatrix}, \begin{bmatrix} 4 & 1 & -3 \end{bmatrix} \}$

11. (a) As above, let $S_1 = \{ \begin{bmatrix} 4 & -1 & 5 \end{bmatrix}, \begin{bmatrix} -2 & 3 & 1 \end{bmatrix} \}$. Find a set T such that $S_1 \subset T$ and T is a basis for R_3.
    (b) As above, let $S_2 = \{ \begin{bmatrix} 4 & 3 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 6 \end{bmatrix} \}$. Either express the vector $\vec{v} = \begin{bmatrix} 5 & -1 & 3 \end{bmatrix}$ as a linear combination of elements in S_2 or show that this is not possible.
    (c) As above, let $S_3 = \{ \begin{bmatrix} -1 & 5 & 3 \end{bmatrix}, \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 & -2 \end{bmatrix} \}$. Either express the vector $\vec{v} = \begin{bmatrix} 5 & -1 & 3 \end{bmatrix}$ as a linear combination of elements in S_2 or show that this is not possible.
    (d) As above, let $S_4 = \{ \begin{bmatrix} 5 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 0 & -2 \end{bmatrix}, \begin{bmatrix} 4 & 1 & -3 \end{bmatrix} \}$. Find a set $T \subset S_4$ so that T is a basis for R_3.

12. Let W be the subspace of P_3 spanned by the set $S = \{t^3 - t^2 + 2t - 1, t^3 - t + 7, t^2 - 3t + 8, t^2 + 2t - 1\}$. Find a basis for W and find the dimension of W.

13. Find a basis for each of the following spaces. Also find the dimension of each space.

    (a) The set of all cubic polynomials of the form $at^3 + bt^2 + ct + d$ satisfying $a - 2b + c = d$.

    (b) The set of all vectors of the form $\begin{bmatrix} a-b+c & 2a-b & b+3c & 2c-5a & a+b+c \end{bmatrix}$

    (c) The set of all symmetric 4×4 matrices.

14. Prove Theorem 4.8

15. Prove Corollary 4.4

16. Given the homogeneous linear system $\begin{cases} x_1 + 2x_2 - x_3 + 4x_4 = 0 \\ 3x_1 - 2x_2 + 5x_3 - x_4 = 0 \\ 4x_1 + x_2 - x_3 = 0 \end{cases}$
    Find a basis for the solution space of this system.

17. Given the homogeneous linear system $\begin{cases} 2x_1 - x_2 - 3x_3 + x_4 = 0 \\ 3x_1 + x_2 - 5x_3 + 2x_4 = 0 \\ x_1 - 3x_2 - x_3 = 0 \end{cases}$
    Find a basis for the solution space of this system.

18. Find a basis for the null space of the matrix $A = \begin{bmatrix} 3 & -1 & 0 & 2 & 4 \\ 4 & 2 & -1 & 0 & 3 \\ 0 & 0 & 2 & 1 & -1 \end{bmatrix}$

19. For each of the following matrices, find all real numbers λ such that the homogeneous system $(\lambda I_n - A)\vec{x} = \vec{0}$ has a nontrivial solution.

    (a) $A = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$

    (b) $A = \begin{bmatrix} 3 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

20. Let $S = \{\begin{bmatrix} 1 & 2 & -4 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -3 & 0 \end{bmatrix}\}$. If $\vec{v} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, find $[\vec{v}]_S$.

21. Let $S = \{t^2 - 3t + 2, t^2 - 4, 2t - 1\}$. If $\vec{v} = t^2 - 2t + 1$, find $[\vec{v}]_S$.

22. Let $S = \{t^2 - 3t + 2, t^2 - 4, 2t - 1\}$ and suppose $[\vec{v}]_S = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$. Find $\vec{v}$.

23. Let $S = \{t^2 - 3t + 2, t^2 - 4, 2t - 1\}$, let $T = \{t^2 - t, t^2 - 1, t + 1\}$, and let $\vec{v} = 2t^2 + t - 3$.

    (a) Find $[\vec{v}]_S$ directly.

    (b) Find $[\vec{v}]_T$ directly.

    (c) Find the transition matrix $P_{S \leftarrow T}$.

    (d) Use the transition matrix $P_{S \leftarrow T}$ to compute $[\vec{v}]_S$.