Math 327
Exam 4 - Practice Problems

1. Let $V = P_3$ and let $S = \{t^3 - 1, t^2 + 1, t + 1\}$. Determine whether or not each of the following are in the span of S:

   (a) $3t^3 + 4t^2 - 2t - 1$

   We begin by translating this problem into matrix form as follows. Notice that each vector in the set S corresponds to a column in the matrix, and that we are looking at the form of an arbitrary element in *span* S:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ -1 & 1 & 1 & d \end{array}\right] \xrightarrow{r_4 + r_1 \to r_4} \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 1 & 1 & a+d \end{array}\right] \xrightarrow{r_4 - r_2 \to r_4} \left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 1 & a+d-b \end{array}\right] \; r_4 + r_1 \to r_4$$

   From this, we see that in order for an element to be in *span* S, it must satisfy the equation $a + d - b = c$ or $a + d = b + c$.

   Notice that for $3t^3 + 4t^2 - 2t - 1$, $a = 3$, $b = 4$, $c = -2$, and $d = -1$. Then $a + d = 2$ and $b + c = 2$, hence $3t^3 + 4t^2 - 2t - 1$ is in *span* S.

   (b) $5t^3 + 3t^2 - 2t - 3$

   Using the same condition as above, notice that for $5t^3 + 3t^2 - 2t - 3$, $a = 5$, $b = 3$, $c = -2$, and $d = -3$. Then $a + d = 2$ and $b + c = 1$, hence $5t^3 + 3t^2 - 2t - 3$ is not in *span* S.

2. Let $V = M_{22}$ and let $S = \left\{ \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$. Determine whether or not each of the following are in the span of S:

   (a) $\begin{bmatrix} 7 & -1 \\ 4 & 5 \end{bmatrix}$

   We will once again use matrices to solve this problem. The general matrix associated with *span* S is as follows:

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & a \\ -1 & 0 & 1 & b \\ 0 & -1 & 1 & c \\ 1 & 2 & 0 & d \end{array}\right] \xrightarrow[r_2 \leftrightarrow r_3]{r_1 \leftrightarrow r_4} \left[\begin{array}{ccc|c} 1 & 2 & 0 & d \\ 0 & -1 & 1 & c \\ -1 & 0 & 1 & b \\ 2 & 1 & 0 & a \end{array}\right] \xrightarrow[r_3 + r_1 \to r_3]{r_4 + 2r_3 \to r_4} \left[\begin{array}{ccc|c} 1 & 2 & 0 & d \\ 0 & -1 & 1 & c \\ 0 & 2 & 1 & b+d \\ 0 & 1 & 2 & a+2b \end{array}\right]$$

$$\xrightarrow[r_4 + r_2 \to r_4]{r_1 - r_3 \to r_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -b \\ 0 & -1 & 1 & c \\ 0 & 2 & 1 & b+d \\ 0 & 0 & 3 & a+2b+c \end{array}\right] \xrightarrow[-r_2 \to r_2]{r_3 + 2r_2 \to r_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -b \\ 0 & 1 & -1 & -c \\ 0 & 0 & 3 & b+2c+d \\ 0 & 0 & 3 & a+2b+c \end{array}\right]$$

   From this, we see that in order to be in *span* S, we must have $b + 2c + d = a + 2b + c$. That is, $a + b = c + d$.

   Notice that for the matrix above, $a = 7$, $b = -1$, $c = 4$ and $d = 5$. So $a + b = 6$ and $c + d = 9$, so this matrix is not in *span* S.

   (b) $\begin{bmatrix} 7 & 1 \\ -1 & 7 \end{bmatrix}$

   Using the same condition as above, notice that for this matrix, $a = 7$, $b = 1$, $c = -1$ and $d = 7$. So $a + b = 8$ and $c + d = 6$, so this matrix is not in *span* S.

3. Which of the following sets span R_3?

   (a) $S = \{\begin{bmatrix} -1 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -2 \end{bmatrix}\}$

   We begin by representing these vectors as a matrix. Notice that each vector is represented by a **column** in the matrix.

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -1 & 3 \\ 5 & 3 & -2 \end{bmatrix}$$

   Since $det(A) = (-2) + (30) + (12) - (-5) - (-16) - (-9) = 70 \neq 0$, then this set of vectors spans R_3.

   (b) $S = \{\begin{bmatrix} 1 & -2 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 1 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 5 & -10 \end{bmatrix}\}$

   We begin by representing these vectors as a matrix. Notice that each vector is represented by a **column** in the matrix.

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -2 & 1 & 5 \\ 4 & -2 & -10 \end{bmatrix}$$

   Since $det(A) = (-10) + (60) + (4) - (4) - (60) - (-10) = 0$, then this set of vectors does not span R_3.

   (c) $S = \{\begin{bmatrix} 2 & -3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 3 \end{bmatrix}, \begin{bmatrix} 2 & -4 & -5 \end{bmatrix}, \begin{bmatrix} 3 & -4 & 3 \end{bmatrix}\}$

   We begin by representing these vectors as a matrix. Notice that each vector is represented by a **column** in the matrix.

$$A = \begin{bmatrix} 2 & 1 & 2 & 3 \\ -3 & -1 & -4 & -4 \\ 1 & 3 & -5 & 3 \end{bmatrix}$$

   Since this matrix is not square and it has more columns than rows, we put this matrix into row echelon form:

$$r_1 \leftrightarrow r_3 \begin{bmatrix} 1 & 3 & -5 & 3 \\ -3 & -1 & -4 & -4 \\ 2 & 1 & 2 & 3 \end{bmatrix} \begin{matrix} \scriptstyle r_2+3r_1 \to r_2 \\ r_3 - 2r_1 \to r_3 \end{matrix} \begin{bmatrix} 1 & 3 & -5 & 3 \\ 0 & -5 & 12 & -3 \\ 0 & 8 & -19 & 5 \end{bmatrix} \begin{matrix} \scriptstyle 2r_3+3r_2 \to r_2 \\ r_3 - 8r_2 \to r_3 \end{matrix} \begin{bmatrix} 1 & 3 & -5 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -3 & -3 \end{bmatrix}$$

$$-\tfrac{1}{3}r_3 \to r_3 \begin{bmatrix} 1 & 3 & -5 & 3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

   Notice that the row echelon form has three leading 1's, so this collection of vectors does span R_3.

4. Describe the subspace of R^3 spanned by the set $S = \{\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}\}$

   We begin by forming a matrix representing an arbitrary linear combination of these vectors, and then we put this matrix into reduced row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 2 & 2 & b \\ -1 & 2 & 1 & c \end{array}\right] \begin{matrix} \scriptstyle r_1+r_3 \to r_3 \\ \frac{1}{2}r_2 \to r_2 \end{matrix} \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & 1 & \frac{b}{2} \\ 0 & 4 & 4 & a+c \end{array}\right] \begin{matrix} \scriptstyle r_1-2r_2 \to r_1 \\ r_3 - 4r_2 \to r_3 \end{matrix} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a-b \\ 0 & 1 & 1 & \frac{b}{2} \\ 0 & 0 & 0 & a+c-2b \end{array}\right]$$

   From this, we see that $spanS = \{\begin{bmatrix} a & b & c \end{bmatrix} : a + c - 2b = 0\}$

5. For each of the following sets of vectors, determine whether or not the set is linearly independent. For those that are dependent, write one of the vectors as a linear combination of the others.

   (a) $S = \{\begin{bmatrix} 2 & -3 \end{bmatrix}, \begin{bmatrix} -1 & 3 \end{bmatrix}, \begin{bmatrix} 2 & -4 \end{bmatrix}\}$

   Notice that the matrix representing the homogeneous system of equations is: $\left[\begin{array}{ccc|c} 2 & -1 & 2 & 0 \\ -3 & 3 & -4 & 0 \end{array}\right]$

   Since this system has more unknowns than equations, it must have at least one free variable, meaning that it has non-trivial solutions. Hence this set is not linearly independent.

Putting this system in reduced row echelon form gives (check the details): $\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \end{array}\right]$

Then, if we take $a_3 = t$, then we have $a_1 = -\frac{2}{3}t$ and $a_2 = \frac{2}{3}t$. If we take $t = 1$, we have $-\frac{2}{3}\vec{v_1} + \frac{2}{3}\vec{v_2} + \vec{v_3} = \vec{0}$, so $\vec{v_3} = \frac{2}{3}\vec{v_1} - \frac{2}{3}\vec{v_2}$.

(b) $S = \{t^2 - t + 1, 2t^2 + 5, 2t + 3\}$

Notice that the matrix representing the homogeneous system of equations is: $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 1 & 5 & 3 & 0 \end{array}\right]$

Putting this system in reduced row echelon form gives (check the details): $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$, so we have a linearly dependent set.

If we take $a_3 = t$, then we have $a_1 = 2t$ and $a_2 = -t$. If we take $t = 1$, we have $2\vec{v_1} - \vec{v_2} + \vec{v_3} = \vec{0}$, so $\vec{v_3} = -2\vec{v_1} + \vec{v_2}$.

(c) $S = \{t^2 - t, 2t^2 + 5, 2t + 3\}$

Notice that the matrix representing the homogeneous system of equations is: $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 5 & 3 & 0 \end{array}\right]$

Putting this system in reduced row echelon form gives (check the details): $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

Then, this system has only the trivial solution, so S is linearly independent.

6. For which values of c are the vectors $t^2 + 2t + 1$, $2 - t$, and $t^2 + t + c^2$ linearly independent?

We begin by representing these vectors as a matrix. Notice that each vector is represented by a **column** in the matrix.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & c^2 \end{bmatrix}$$

Since $det(A) = (-c^2) + (0) + (4) - (-1) - (0) - (2) = -c^2 + 3$, then $det(A) = 0$ if and only if $c = \pm\sqrt{3}$. Hence these vectors are linearly independent provided $c \neq \pm\sqrt{3}$.

7. Suppose that $S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly independent. Let $T = \{\vec{w_1}, \vec{w_2}, \vec{w_3}\}$, where $\vec{w_1} = \vec{v_1} - \vec{v_2}$, $\vec{w_2} = \vec{v_1} + \vec{v_2} + \vec{v_3}$, and $\vec{w_3} = 2\vec{v_1} + \vec{v_3}$. Determine whether or not T is linearly independent.

Suppose that $a_1\vec{w_1} + a_2\vec{w_2} + a_3\vec{w_3} = \vec{0}$. Then, substituting, $a_1(\vec{v_1} - \vec{v_2}) + a_2(\vec{v_1} + \vec{v_2} + \vec{v_3}) + a_3(2\vec{v_1} + \vec{v_3}) = \vec{0}$.

Expanding out and then combining terms gives: $a_1\vec{v_1} - a_1\vec{v_2} + a_2\vec{v_1} + a_2\vec{v_2} + a_2\vec{v_3} + 2a_3\vec{v_1} + a_3\vec{v_3} = \vec{0}$, so $(a_1 + a_2 + 2a_3)\vec{v_1} + (-a_1 + a_2)\vec{v_2} + (a_2 + a_3)\vec{v_3} = \vec{0}$.

This homogeneous system of linear equations is represented by the matrix: $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

Notice that $det(A) = 1 + 0 + -2 - (0) - (-1) - (0) = 1 - 2 + 1 = 0$. Hence this system has non-trivial solutions, and therefore, T is not linearly independent. (In fact, one can verify that if we take $a_1 = -1$, $a_2 = -1$ and $a_3 = 1$, then $a_1\vec{w_1} + a_2\vec{w_2} + a_3\vec{w_3} = \vec{0}$.

8. Prove the following: If $S = \{\vec{v_1}, \vec{v_2}\}$ is a linearly independent set and $\vec{v_3}$ is not in the span of S, then $T = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly independent.

Proof: Since we wish to show that T is linearly independent, suppose that $a_1\vec{v_1} + a_2\vec{v_2} + a_3\vec{v_3} = \vec{0}$.

Case 1: If $a_3 = 0$, then this becomes $a_1\vec{v_1} + a_2\vec{v_2} + (0)\vec{v_3} = a_1\vec{v_1} + a_2\vec{v_2} = \vec{0}$. Therefore, since $S = \{\vec{v_1}, \vec{v_2}\}$ is a linearly independent set, then we must have $a_1 = a_2 = 0$. Then $a_1 = a_2 = a_3 = 0$

Case 2: If $a_3 \neq 0$, then $a_1\vec{v_1} + a_2\vec{v_2} = -a_3\vec{v_3}$, so $\vec{v_3} = -\frac{a_1}{a_3}\vec{v_1} - \frac{a_2}{a_3}\vec{v_2}$. But this contradicts the fact that $\vec{v_3}$ is not in the span of $S = \{\vec{v_1}, \vec{v_2}\}$. So this case is not possible.

Hence $T = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is linearly independent. □.

9. Let S_1 and S_2 be finite subsets of a vector space V. Suppose that $S_1 \subset S_2$ and that S_1 is linearly independent. Must S_2 be linearly independent?

No. Adding a vector to a linearly independent set could create a linear dependence. For example, suppose $S_1 = \{\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}\}$ and $S_2 = \{\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix}\}$.
Then S_1 is linearly independent (in fact, it is the standard basis for R_2), but S_2 is linearly dependent, since $\vec{v_1} + \vec{v_2} - \vec{v_3} = \vec{0}$.

10. Determine which of the following sets form a basis for R_3. For those that are not, state which part(s) of the definition of a basis are not satisfied.

(a) $S_1 = \{\begin{bmatrix} 4 & -1 & 5 \end{bmatrix}, \begin{bmatrix} -2 & 3 & 1 \end{bmatrix}\}$

We know that $dim(R_3) = 3$, so this set is not a basis. Alternatively, we can see from the associated matrix: $\left[\begin{array}{cc|c} 4 & -2 & a \\ -1 & 3 & b \\ 5 & 1 & c \end{array}\right]$ that this set of vectors does not span R_3 and hence is not a basis for R_3. Notice that since these two vectors are not multiples of each other, this set is linearly independent.

(b) $S_2 = \{\begin{bmatrix} 4 & 3 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 6 \end{bmatrix}\}$

We consider the associated matrix: $M = \begin{bmatrix} 4 & 2 & 2 \\ 3 & 3 & 0 \\ 5 & -1 & 6 \end{bmatrix}$.

Notice that $det(M) = 72 + 0 - 6 - 30 - 36 - 0 = 0$. Therefore, this set is linearly dependent, and hence is not a basis for R_3. In addition, since removing one vector that is a linear combination of the others will leave us with only two vectors, this set also does not span R_3.

(c) $S_3 = \{\begin{bmatrix} -1 & 5 & 3 \end{bmatrix}, \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 & -2 \end{bmatrix}\}$

We consider the associated matrix: $M = \begin{bmatrix} -1 & 3 & 2 \\ 5 & -2 & 3 \\ 3 & 1 & -2 \end{bmatrix}$.

Notice that $det(M) = -4 + 27 + 10 - (-12) - (-30) - (-3) = 78$. Therefore, this set is a basis for R_3.

(d) $S_4 = \{\begin{bmatrix} 5 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 0 & -2 \end{bmatrix}, \begin{bmatrix} 4 & 1 & -3 \end{bmatrix}\}$

We consider the associated matrix: $M = \begin{bmatrix} 5 & 2 & 3 & 4 \\ 3 & -1 & 0 & 1 \\ -1 & 4 & -2 & -3 \end{bmatrix}$.

Using appropriate row operations (verify this), we find that the reduced row echelon form of this matrix is: $\begin{bmatrix} 1 & 0 & 0 & \frac{3}{11} \\ 0 & 1 & 0 & -\frac{2}{11} \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

From this, we see that this set of vectors does span R^3, but since there is a free variable, it is linearly dependent, so this set is not a basis for R_3.

11. (a) As above, let $S_1 = \{\begin{bmatrix} 4 & -1 & 5 \end{bmatrix}, \begin{bmatrix} -2 & 3 & 1 \end{bmatrix}\}$. Find a set T such that $S_1 \subset T$ and T is a basis for R_3.

As mentioned above, since $dim(R^3) = 3$, we must add a third vector that is linearly independent from the previous two in order to obtain a basis for R_3.

We will try adding the vector $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$.

We consider the associated matrix: $M = \begin{bmatrix} 4 & -2 & 0 \\ -1 & 3 & 0 \\ 5 & 1 & 1 \end{bmatrix}$.

Notice that $det(M) = 12+0+0-(0)-(2)-(0) = 10$. Therefore, the set $T = \{\begin{bmatrix} 4 & -1 & 5 \end{bmatrix}, \begin{bmatrix} -2 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\}$ is a basis for R^3.

(b) As above, let $S_2 = \{\begin{bmatrix} 4 & 3 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 6 \end{bmatrix}\}$. Either express the vector $\vec{v} = \begin{bmatrix} 5 & -1 & 3 \end{bmatrix}$ as a linear combination of elements in S_2 or show that this is not possible.

We consider the associated matrix: $M = \left[\begin{array}{ccc|c} 4 & 2 & 2 & 5 \\ 3 & 3 & 0 & -1 \\ 5 & -1 & 6 & 3 \end{array}\right]$.

Notice that the reduced row echelon form for this matrix is (verify this): $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$. Therefore, the vector $\vec{v} = \begin{bmatrix} 5 & -1 & 3 \end{bmatrix}$ cannot be written as a linear combination of elements in S_2.

(c) As above, let $S_3 = \{\begin{bmatrix} -1 & 5 & 3 \end{bmatrix}, \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 & -2 \end{bmatrix}\}$. Either express the vector $\vec{v} = \begin{bmatrix} 5 & -1 & 3 \end{bmatrix}$ as a linear combination of elements in S_3 or show that this is not possible.

As shown above, S_3 is a basis for R_3, so any vector can be written as a linear combination of elements in S_3.

(d) As above, let $S_4 = \{\begin{bmatrix} 5 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 0 & -2 \end{bmatrix}, \begin{bmatrix} 4 & 1 & -3 \end{bmatrix}\}$. Find a set $T \subset S_4$ so that T is a basis for R_3.

In problem 10(d) above, we observed that the reduced row echelon form for the matrix associated with this set has the form: $\begin{bmatrix} 1 & 0 & 0 & \frac{3}{11} \\ 0 & 1 & 0 & -\frac{2}{11} \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

From this, we see that taking the first three vectors in the original set (those corresponding to the leading 1's in the reduced form) gives a basis for R_3. That is, the set $T = \{\begin{bmatrix} 5 & 3 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 0 & -2 \end{bmatrix}\}$ is a basis for R_3.

12. Let W be the subspace of P_3 spanned by the set $S = \{t^3 - t^2 + 2t - 1, t^3 - t + 7, t^2 - 3t + 8, t^2 + 2t - 1\}$. Find a basis for W and find the dimension of W.

We begin by looking at the related matrix: $\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 2 & -1 & -3 & 2 & 0 \\ -1 & 7 & 8 & -1 & 0 \end{array}\right]$

After performing the appropriate row operations to put this matrix into reduced row echelon form, we have:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Noting the placement and number of the leading 1's, we see that $dim\, W = 3$, and that $t = \{t^3 - t^2 + 2t - 1, t^3 - t + 7, t^2 + 2t - 1\}$ is a basis for W.

13. Find a basis for each of the following spaces. Also find the dimension of each space.

(a) The set of all cubic polynomials of the form $at^3 + bt^2 + ct + d$ satisfying $a - 2b + c = d$.

Since $a-2b+c = d$, these polynomials are all of the form: $at^3+bt^2+ct+(a-2b+c) = a(t^3+1)+b(t^2-2)+c(t+1)$.

Therefore, the set $S = \{t^3 + 1, t^2 - 2, t + 1\}$ spans this set of cubic polynomials.

The matrix representing the homogeneous system of equations that checks for linear independence is: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 \end{array}\right]$,

which we can see has only the trivial solution. So this set is linearly independent and hence is a basis for the given set of cubic polynomials.

(b) The set of all vectors of the form $\left[\begin{array}{ccccc} a-b+c & 2a-b & b+3c & 2c-5a & a+b+c \end{array}\right]$

Notice that each of these vectors can be expressed as:

$a\left[\begin{array}{ccccc} 1 & 2 & 0 & -5 & 1 \end{array}\right] + b\left[\begin{array}{ccccc} -1 & -1 & 1 & 0 & 1 \end{array}\right] + c\left[\begin{array}{ccccc} 1 & 0 & 3 & 2 & 1 \end{array}\right]$.

Therefore, $S = \left\{\left[\begin{array}{ccccc} 1 & 2 & 0 & -5 & 1 \end{array}\right], \left[\begin{array}{ccccc} -1 & -1 & 1 & 0 & 1 \end{array}\right], \left[\begin{array}{ccccc} 1 & 0 & 3 & 2 & 1 \end{array}\right]\right\}$ is a spanning set for this collection of matrices.

The matrix representing the homogeneous system of equations that checks for linear independence is:

$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -5 & 1 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 2 & 1 & 0 \end{array}\right]$, which is row equivalent to the following matrix (check this...)

$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & \frac{19}{5} & -\frac{7}{5} & 0 \\ 0 & 1 & 0 & -\frac{22}{5} & \frac{6}{5} & 0 \\ 0 & 0 & 1 & -\frac{3}{5} & \frac{4}{5} & 0 \end{array}\right]$. Hence this set is linearly independent as well, and hence is a basis for the given set of vectors.

(c) The set of all symmetric 4×4 matrices.

Recall that symmetric 4×4 matrices must be of the form: $\begin{bmatrix} a & b & d & f \\ b & c & e & h \\ d & e & g & i \\ f & h & i & j \end{bmatrix}$

Then the following set spans the set of symmetric 4×4 matrices:

$$\left\{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left.\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right\}$$

We claim that this set is also linearly independent.

14. Prove Theorem 4.8

Theorem 4.8: If $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ is a basis for a vector space V, then every vector V can be written in one and only one way as a linear combination of the vectors in S.

Proof:

Let $\vec{v} \in V$. Since S is a basis, it spans V. Thus $\vec{v} = a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n}$ for some constants $a_i \in \mathbb{R}$.

Suppose that $\vec{v} = b_1\vec{v_1} + b_2\vec{v_2} + \cdots + b_n\vec{v_n}$ is another linear combination representing $\vec{v}$.

Then, subtracting these, $\vec{v} - \vec{v} = \vec{0} = (a_1 - b_1)\vec{v_1} + (a_2 - b_2)\vec{v_2} + \cdots + (a_n - b_n)\vec{v_n}$

Since S is a basis for V, S is a linearly independent set. Therefore we must have $a_i - b_i = 0$ for each $i = 1 \cdots n$. That is, $a_i = b_i$ for all $i = 1 \cdots n$. Hence the linear combination representing $\vec{v}$ is unique. □.

15. Prove Corollary 4.4

Corollary 4.4: If a vector space V has dimension n, then any subset of $m > n$ vectors must be linearly independent.

Proof:

Suppose that V has dimension n. Then V has a basis S consisting of exactly n vectors. Let m be a positive integer with $m > n$. Let T be a set of m distinct vectors drawn from V. Suppose that T is linearly independent. According to Theorem 4.10, any linearly independent set of vectors must satisfy $m \leq n$. This contradicts out previous assumption that $m > n$. Hence T cannot be linearly independent. This T is linearly dependent. □

Let $\vec{v} \in V$. Since S is a basis, it spans V. Thus $\vec{v} = a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_n\vec{v_n}$ for some constants $a_i \in \mathbb{R}$.

Suppose that $\vec{v} = b_1\vec{v_1} + b_2\vec{v_2} + \cdots + b_n\vec{v_n}$ is another linear combination representing $\vec{v}$.

Then, subtracting these, $\vec{v} - \vec{v} = \vec{0} = (a_1 - b_1)\vec{v_1} + (a_2 - b_2)\vec{v_2} + \cdots + (a_n - b_n)\vec{v_n}$

Since S is a basis for V, S is a linearly independent set. Therefore we must have $a_i - b_i = 0$ for each $i = 1 \cdots n$. That is, $a_i = b_i$ for all $i = 1 \cdots n$. Hence the linear combination representing $\vec{v}$ is unique.

16. Given the homogeneous linear system $\begin{cases} x_1 + 2x_2 - x_3 + 4x_4 = 0 \\ 3x_1 - 2x_2 + 5x_3 - x_4 = 0 \\ 4x_1 + x_2 - x_3 = 0 \end{cases}$
Find a basis for the solution space of this system.

We begin by expressing this system as a matrix, which we then put into reduced row echelon form:

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 0 \\ 3 & -2 & 5 & -1 & 0 \\ 4 & 1 & -1 & 0 & 0 \end{array}\right] \xrightarrow[r_3 - 4r_1 \to r_3]{r_2 - 3r_1 \to r_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 0 \\ 0 & -8 & 8 & -13 & 0 \\ 0 & -7 & 3 & -6 & 0 \end{array}\right] \xrightarrow[r_3 + 7r_2 \to r_3]{-r_2 + r_3 \to r_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 0 \\ 0 & 1 & -5 & -3 & 0 \\ 0 & 0 & -32 & -37 & 0 \end{array}\right] \xrightarrow[-\frac{1}{32}r_3 \to r_3]{r_1 - 2r_2 \to r_1}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 9 & 10 & 0 \\ 0 & 1 & -5 & -3 & 0 \\ 0 & 0 & 1 & \frac{37}{32} & 0 \end{array}\right] \xrightarrow[r_2 + 5r_3 \to r_2]{r_1 - 9r_3 \to r_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{13}{32} & 0 \\ 0 & 1 & 0 & \frac{89}{32} & 0 \\ 0 & 0 & 1 & \frac{37}{32} & 0 \end{array}\right]$$

From this, we have one free variable, $x_4 = t$. Then $x_1 = \frac{13}{32}t$, $x_2 = -\frac{89}{32}t$, and $x_3 = -\frac{37}{32}t$

Hence, all solutions are of the form: $\begin{bmatrix} \frac{13}{32}t \\ -\frac{89}{32}t \\ \frac{37}{32}t \\ t \end{bmatrix}$. Thus the set $S = \left\{ \begin{bmatrix} \frac{13}{32} \\ -\frac{89}{32} \\ \frac{37}{32} \\ 1 \end{bmatrix} \right\}$ is a basis for the solution space.

17. Given the homogeneous linear system $\begin{cases} 2x_1 - x_2 - 3x_3 + x_4 = 0 \\ 3x_1 + x_2 - 5x_3 + 2x_4 = 0 \\ x_1 - 3x_2 - x_3 = 0 \end{cases}$
Find a basis for the solution space of this system.

We begin by expressing this system as a matrix:

$$\left[\begin{array}{cccc|c} 2 & -1 & -3 & 1 & 0 \\ 3 & 1 & -5 & 2 & 0 \\ 1 & -3 & -1 & 0 & 0 \end{array}\right]$$

After carrying out the appropriate row operations (check the details), we see that the reduced row echelon form for this matrix is:

$$\left[\begin{array}{cccc|c} 1 & 0 & -\frac{8}{5} & \frac{3}{5} & 0 \\ 0 & 1 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

From this, we have two free variables, $x_3 = s$ and $x_4 = t$. Then $x_1 = \frac{8}{5}s - \frac{3}{5}t$ and $x_2 = \frac{1}{5}s - \frac{1}{5}t$

Hence, all solutions are of the form: $\begin{bmatrix} \frac{8}{5}s - \frac{3}{5}t \\ \frac{1}{5}s - \frac{1}{5}t \\ s \\ t \end{bmatrix}$. Thus the set $S = \left\{ \begin{bmatrix} \frac{8}{5} \\ \frac{1}{5} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{5} \\ -\frac{1}{5} \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the solution space.

18. Find a basis for the null space of the matrix $A = \begin{bmatrix} 3 & -1 & 0 & 2 & 4 \\ 4 & 2 & -1 & 0 & 3 \\ 0 & 0 & 2 & 1 & -1 \end{bmatrix}$

We consider the matrix for the related homogeneous system of equations:

$$\left[\begin{array}{ccccc|c} 3 & -1 & 0 & 2 & 4 & 0 \\ 4 & 2 & -1 & 0 & 3 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 \end{array}\right]$$

After carrying out the appropriate row operations (check the details), we see that the reduced row echelon form for this matrix is:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & \frac{9}{20} & \frac{21}{20} & 0 \\ 0 & 1 & 0 & -\frac{13}{20} & -\frac{17}{20} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

From this, we have two free variables, $x_4 = s$ and $x_5 = t$. Then $x_1 = -\frac{9}{20}s - \frac{21}{20}t$, $x_2 = \frac{13}{20}s + \frac{17}{20}t$, and $x_3 = -\frac{1}{2}s + \frac{1}{2}t$

Hence, all solutions are of the form: $\begin{bmatrix} -\frac{9}{20}s - \frac{21}{20}t \\ \frac{13}{20}s + \frac{17}{20}t \\ -\frac{1}{2}s + \frac{1}{2}t \\ s \\ t \end{bmatrix}$. Thus the set $S = \left\{ \begin{bmatrix} -\frac{9}{20} \\ \frac{13}{20} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{21}{20} \\ \frac{17}{20} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the solution space.

19. For each of the following matrices, find all real numbers λ such that the homogeneous system $(\lambda I_n - A)\vec{x} = \vec{0}$ has a nontrivial solution.

(a) $A = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$

Let $M = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} \lambda - 3 & -2 \\ -2 & \lambda + 1 \end{bmatrix}$.

Then there is a nontrivial solution when $det(M) = (\lambda - 3)(\lambda + 1) - 4 = \lambda^2 - 2\lambda - 7 = 0$.

Using the quadratic formula, we see that non-trivial solutions exist if and only if $\lambda = \frac{2 \pm \sqrt{4 - 4(1)(-7)}}{2(1)} = \frac{2 \pm \sqrt{32}}{2} = 1 \pm 2\sqrt{2}$.

(b) $A = \begin{bmatrix} 3 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Let $M = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda - 3 & 0 & -2 \\ -1 & \lambda + 1 & 0 \\ 0 & 0 & \lambda - 1 \end{bmatrix}$

Then there is a nontrivial solution when $det(M) = (\lambda - 3)(\lambda + 1)(\lambda - 1) + 0 + 0 - 0 - 0 - 0 = 0$.

Therefore, we see that non-trivial solutions exist if and only if $\lambda = 3$, $\lambda = -1$, or $\lambda = 1$.

20. Let $S = \{ \begin{bmatrix} 1 & 2 & -4 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -3 & 0 \end{bmatrix} \}$. If $\vec{v} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, find $[\vec{v}]_S$.

We find the coordinate of $\vec{v}$ with respect to the ordered basis S by considering the following matrix:

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 2 & -1 & -3 & 2 \\ -4 & 2 & 0 & 3 \end{array}\right]$$

The reduced row echelon form of this matrix is (check this by carrying out row operations): $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{6} \\ 0 & 1 & 0 & \frac{7}{6} \\ 0 & 0 & 1 & -\frac{7}{6} \end{array}\right]$

Hence $[\vec{v}]_S = \begin{bmatrix} -\frac{1}{6} \\ \frac{7}{6} \\ -\frac{7}{6} \end{bmatrix}$

21. Let $S = \{t^2 - 3t + 2, t^2 - 4, 2t - 1\}$. If $\vec{v} = t^2 - 2t + 1$, find $[\vec{v}]_S$.

We find the coordinate of $\vec{v}$ with respect to the ordered basis S by considering the following matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ -3 & 0 & 2 & -2 \\ 2 & -4 & -1 & 1 \end{array}\right]$$

The reduced row echelon form of this matrix is (check this by carrying out row operations): $\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{8}{9} \\ 0 & 1 & 0 & \frac{1}{9} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right]$

Hence $[\vec{v}]_S = \begin{bmatrix} \frac{8}{9} \\ \frac{1}{9} \\ \frac{1}{3} \end{bmatrix}$

22. Let $S = \{t^2 - 3t + 2, t^2 - 4, 2t - 1\}$ and suppose $[\vec{v}]_S = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$. Find $\vec{v}$.

Using the coordinates with respect to the ordered bases S, $\vec{v} = 3(t^2 - 3t + 2) - 2(t^2 - 4) + (2t - 1) = 3t^2 - 9t + 6 - 2t^2 + 8 + 2t - 1 = t^2 - 7t + 13$.

23. Let $S = \{t^2 - 3t + 2, t^2 - 4, 2t - 1\}$, let $T = \{t^2 - t, t^2 - 1, t + 1\}$, and let $\vec{v} = 2t^2 + t - 3$.

(a) Find $[\vec{v}]_S$ directly.

Notice that the matrix associated with this computation is: $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ -3 & 0 & 2 & 1 \\ 2 & -4 & -1 & -3 \end{array}\right]$

The reduced row echelon form of this matrix is (check this by carrying out row operations): $\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{11}{9} \\ 0 & 1 & 0 & \frac{7}{9} \\ 0 & 0 & 1 & \frac{7}{3} \end{array}\right]$

Hence $[\vec{v}]_S = \begin{bmatrix} \frac{11}{9} \\ \frac{7}{9} \\ \frac{7}{3} \end{bmatrix}$

(b) Find $[\vec{v}]_T$ directly.

Notice that the matrix associated with this computation is: $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -3 \end{array}\right]$

The reduced row echelon form of this matrix is (check this by carrying out row operations): $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right]$

Hence $[\vec{v}]_T = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$

(c) Find the transition matrix $P_{S\leftarrow T}$.

To find $P_{S\leftarrow T}$, we begin with a partitioned matrix. The columns of the left half correspond to the vectors in S. The columns in the right half correspond to the vectors in T.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & 0 \\ -3 & 0 & 2 & -1 & 0 & 1 \\ 2 & -4 & -1 & 0 & -1 & 1 \end{array}\right]$$

The reduced row echelon form of this matrix is (check this by carrying out row operations): $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{9} & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} & 1 & 1 \end{array}\right]$

Hence $P_{S\leftarrow T} = \begin{bmatrix} \frac{7}{9} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{9} & \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & 1 & 1 \end{bmatrix}$

(d) Use the transition matrix $P_{S\leftarrow T}$ to compute $[\vec{v}]_S$.

Recall that $[\vec{v}]_S = P_{S\leftarrow T}[\vec{v}]_T = \begin{bmatrix} \frac{7}{9} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{9} & \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{11}{9} \\ \frac{7}{9} \\ \frac{7}{3} \end{bmatrix}$.