Math 327
Exam 5 - Practice Problem Solutions

1. Find a basis for the row space of each of the following matrices. Your basis should consist of rows of the original matrix.

   (a) $\begin{bmatrix} 1 & -2 & 5 & 4 \\ 2 & 1 & -3 & 7 \\ 1 & -7 & 18 & 5 \end{bmatrix}$

   Since we want a basis for the row space consisting of rows of the original matrix, we will take A^T and put it into reduced row echelon form. The position of the 1's in the reduced row echelon form indicate the rows of A that form a basis for the row space. We omit the details of the row reduction process.

$$A^T = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & -7 \\ 5 & -3 & 18 \\ 4 & 7 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   From the reduced form above, the following is a basis for the row space: $\{\begin{bmatrix} 1 & -2 & 5 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 1 & -3 & 7 \end{bmatrix}\}$

   (b) $\begin{bmatrix} 1 & 3 & -4 & 1 \\ 0 & 4 & -5 & 2 \\ 2 & -1 & 1 & 5 \\ 2 & 3 & -4 & 7 \\ 3 & 2 & -3 & 6 \end{bmatrix}$

   Since we want a basis for the row space consisting of rows of the original matrix, we will take A^T and put it into reduced row echelon form. The position of the 1's in the reduced row echelon form indicate the rows of A that form a basis for the row space. We omit the details of the row reduction process.

$$A^T = \begin{bmatrix} 1 & 0 & 2 & 2 & 3 \\ 3 & 4 & -1 & 3 & 2 \\ -4 & -5 & 1 & -4 & -3 \\ 1 & 2 & 5 & 7 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   From the reduced form above, the following is a basis for the row space:

   $\{\begin{bmatrix} 1 & 3 & -4 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 4 & -5 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 & 1 & 5 \end{bmatrix}\}$

2. Find a basis for the column space of each of the matrices from problem 1. Your basis should consist of columns of the original matrix.

   (a) Since we want a basis for the column space consisting of columns of the original matrix, we will take A and put it into reduced row echelon form. The position of the 1's in the reduced row echelon form indicate the columns of A that form a basis for the row space. We omit the details of the row reduction process.

$$A = \begin{bmatrix} 1 & -2 & 5 & 4 \\ 2 & 1 & -3 & 7 \\ 1 & -7 & 18 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{5} & \frac{18}{5} \\ 0 & 1 & -\frac{13}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   From the reduced form above, the following is a basis for the column space: $\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -7 \end{bmatrix}\right\}$

   (b) Since we want a basis for the column space consisting of columns of the original matrix, we will take A and put it into reduced row echelon form. The position of the 1's in the reduced row echelon form indicate the columns of A that form a basis for the row space. We omit the details of the row reduction process.

$$A = \begin{bmatrix} 1 & 3 & -4 & 1 \\ 0 & 4 & -5 & 2 \\ 2 & -1 & 1 & 5 \\ 2 & 3 & -4 & 7 \\ 3 & 2 & -3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 33 \\ 0 & 0 & 1 & 26 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From the reduced form above, the following is a basis for the column space: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -5 \\ 1 \\ -4 \\ -3 \end{bmatrix} \right\}$

3. Find the rank of each of the matrices from problem 1.

   (a) From the work done above, we see that $rank\, A = 2$.

   (b) From the work done above, we see that $rank\, A = 3$.

4. Find a basis for the null space of each of the matrices from problem 1.

   (a) To find a basis for the null space, we need to row reduce the matrix representing the related homogeneous system of equations. Since we already put A in reduced row echelon form, we can just add an augmented column of zeros to the reduced row echelon form of A:

$$\left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{5} & \frac{18}{5} & 0 \\ 0 & 1 & -\frac{13}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

   From this, we see that $x_1 = \frac{1}{5}x_3 - \frac{18}{5}x_4$ and $x_2 = \frac{13}{5}x_3 + \frac{1}{5}x_4$. Let $x_3 = s$ and $x_4 = t$.

   Then elements of the null space have the form: $\begin{bmatrix} \frac{1}{5}s - \frac{18}{5}t \\ \frac{13}{5}s + \frac{1}{5}t \\ s \\ t \end{bmatrix}$. Therefore, the following set is a basis for the null space: $\left\{ \begin{bmatrix} \frac{1}{5} \\ \frac{13}{5} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{18}{5} \\ \frac{1}{5} \\ 0 \\ 1 \end{bmatrix} \right\}$

   (b) find a basis for the null space, we need to row reduce the matrix representing the related homogeneous system of equations. Since we already put A in reduced row echelon form, we can just add an augmented column of zeros to the reduced row echelon form of A:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 6 & 0 \\ 0 & 1 & 0 & 33 & 0 \\ 0 & 0 & 1 & 26 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

   From this, we see that $x_1 = -6x_4$ and $x_2 = -33x_4$ and $x_3 = -26x_4$. Let $x_4 = t$.

   Then elements of the null space have the form: $\begin{bmatrix} -6t \\ -33t \\ -26t \\ t \end{bmatrix}$. Therefore, the following set is a basis for the null space: $\left\{ \begin{bmatrix} -6 \\ -33 \\ -26 \\ 1 \end{bmatrix} \right\}$

5. Use the determinant to determine whether or not each of the following matrices has rank 3:

   (a) $\begin{bmatrix} -2 & 3 & 1 \\ 0 & -1 & 2 \\ 2 & 1 & 6 \end{bmatrix}$

   Notice that $det(A) = 12 + 12 + 0 - (-2) - 0 - (-4) = 12 + 12 + 2 + 4 = 30$.

   Since $det(A) \neq 0$, A is invertible, and hence reduces to I_3. Thus $rank\, A = 3$.

(b) $\begin{bmatrix} -2 & 3 & 1 \\ 0 & 4 & 7 \\ 2 & 1 & 6 \end{bmatrix}$

Notice that $det(A) = -48 + 42 + 0 - (8) - 0 - (-14) = -48 + 42 - 8 + 14 = 0$.

Since $det(A) = 0$, A is singular and its reduced form has at least one row of zeros. Thus $rank\, A < 3$.

6. Use the rank of the coefficient matrix to determine whether or not each of the following homogeneous systems have a non-trivial solution.

(a) $A\vec{x} = \vec{0}$ if $A = \begin{bmatrix} 5 & -1 & 2 & 4 \\ 2 & -3 & 0 & 1 \\ 1 & -5 & 3 & 7 \\ 0 & 2 & -3 & 1 \end{bmatrix}$

We find the rank of A by putting A into reduced row echelon form:

$$A = \begin{bmatrix} 5 & -1 & 2 & 4 \\ 2 & -3 & 0 & 1 \\ 1 & -5 & 3 & 7 \\ 0 & 2 & -3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice that A has rank 4, so A has nullity 0. That is, the related homogeneous system of equations has only the trivial solution.

(b) $A\vec{x} = \vec{0}$ if $A = \begin{bmatrix} 5 & -1 & 2 & 4 \\ 2 & -3 & 0 & 1 \\ 1 & -5 & 3 & 7 \\ 0 & -7 & 6 & 13 \end{bmatrix}$

We find the rank of A by putting A into reduced row echelon form:

$$A = \begin{bmatrix} 5 & -1 & 2 & 4 \\ 2 & -3 & 0 & 1 \\ 1 & -5 & 3 & 7 \\ 0 & -7 & 6 & 13 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{53} \\ 0 & 1 & 0 & -\frac{17}{53} \\ 0 & 0 & 1 & \frac{95}{53} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that A has rank 3, so A has nullity 1. That is, there are non-trivial solutions to the related homogeneous system of equations.

7. Suppose that A is a 4×7 matrix.

(a) What is the maximum rank of A?

Since A has 4 rows and 7 columns, the largest identity matrix that fits inside A is I_4. Therefore, the maximum rank of A is 4.

(b) Could the columns of A be linearly independent? Justify your answer.

No. Since there are 7 columns and the maximum rank of A is 4, some of the columns must be linear combinations of the other columns.

(c) Could the rows of A be linearly independent? Justify your answer.

Yes. If A attains its maximum rank of 4, then the rows would be linearly independent.

(d) If the rank of A is 3, find the nullity of A.

If the rank of A is 3, then the reduced row echelon form of A has three 1's. Then there must be 4 free variables, so the nullity of A is 4.

(e) If the rank of A is 3, find the nullity of A^T.

If the rank of A is 3, then the rank of A^T is also 3. Then the reduced row echelon form of A^T also has three 1's. Since A^T has 7 rows and 4 columns, there is one free variable, so A^T has nullity 1.

8. Let A be an $n \times n$ matrix. Show that rank $A = n$ if and only if the columns of A are linearly independent.

Let A be an $n \times n$ matrix and first suppose that $rank\, A = n$. Then A is row equivalent to I_n. Therefore, every column of A is a vector in a basis for the column space of A. Thus the columns of A are linearly independent.

Now, let A be an $n \times n$ matrix and suppose that the columns of A are linearly independent. By definition, the columns of A span the column space of A. Since these columns are linearly independent, the n columns of A form a basis for the column space of A. Then A has column rank n. However, the column rank of A is equal to the rank of A. Thus A has rank n.

9. Consider the following functions from $R^3 \to R^2$:

$$L_1\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 - u_2 \\ u_3 \end{bmatrix} \qquad L_2\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 - u_2 \\ 0 \end{bmatrix}$$

$$L_3\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 - u_2 \\ 1 \end{bmatrix} \qquad L_4\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 - u_2 \\ u_3^2 \end{bmatrix}$$

(a) Determine whether or not each of these functions is a linear transformation. Justify your answer.

i. $L_1\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 - u_2 \\ u_3 \end{bmatrix}$

Notice that $L_1(\vec{u})+L_1(\vec{v}) = L_1\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right)+L_1\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 - u_2 \\ u_3 \end{bmatrix}+\begin{bmatrix} v_1 - v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1 - u_2 + v_1 - v_2 \\ u_3 + v_3 \end{bmatrix} =$
$\begin{bmatrix} (u_1 + v_1) - (u_2 + v_2) \\ (u_3 + v_3) \end{bmatrix}$

Similarly, $L_1(\vec{u} + \vec{v}) = L_1\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}\right) = \begin{bmatrix} (u_1 + v_1) - (u_2 + v_2) \\ (u_3 + v_3) \end{bmatrix} = L_1(\vec{u}) + L_1(\vec{v})$. Hence property (a) holds.

Next, notice that $L_1(c \cdot \vec{u}) = L_1\left(\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}\right) = \begin{bmatrix} cu_1 - cu_2 \\ cu_3 \end{bmatrix} = c \cdot \begin{bmatrix} u_1 - u_2 \\ u_3 \end{bmatrix} = c \cdot L_1(\vec{u})$. Thus property (b) also holds. Therefore, L_1 is a linear transformation.

ii. $L_2\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 - u_2 \\ 0 \end{bmatrix}$

Notice that $L_2(\vec{u})+L_2(\vec{v}) = L_2\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right)+L_2\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 - u_2 \\ 0 \end{bmatrix}+\begin{bmatrix} v_1 - v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 - u_2 + v_1 - v_2 \\ 0 + 0 \end{bmatrix} =$
$\begin{bmatrix} (u_1 + v_1) - (u_2 + v_2) \\ 0 \end{bmatrix}$

Similarly, $L_2(\vec{u} + \vec{v}) = L_2\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}\right) = \begin{bmatrix} (u_1 + v_1) - (u_2 + v_2) \\ 0 \end{bmatrix} = L_2(\vec{u}) + L_2(\vec{v})$. Hence property (a) holds.

Next, notice that $L_2(c \cdot \vec{u}) = L_2\left(\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}\right) = \begin{bmatrix} cu_1 - cu_2 \\ 0 \end{bmatrix} = c \cdot \begin{bmatrix} u_1 - u_2 \\ 0 \end{bmatrix} = c \cdot L_2(\vec{u})$. Thus property (b) also holds. Therefore, L_2 is a linear transformation.

iii. $L_3\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 - u_2 \\ 1 \end{bmatrix}$

Notice that $L_3\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}\neq\begin{bmatrix}0\\0\end{bmatrix}$

By Theorem 6.1(a), the image of $\vec{0}$ under any linear transformation must be the zero vector. Hence L_3 is not a linear transformation.

iv. $L_4\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\right)=\begin{bmatrix}u_1-u_2\\u_3^2\end{bmatrix}$

Notice that $3\cdot L_4\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)=3\cdot\begin{bmatrix}1-1\\1^2\end{bmatrix}=3\cdot\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}$

On the other hand, $L_4\left(3\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}\right)=L_4\left(\begin{bmatrix}3\\3\\3\end{bmatrix}\right)=\begin{bmatrix}3-3\\3^2\end{bmatrix}=\begin{bmatrix}0\\9\end{bmatrix}\neq\begin{bmatrix}0\\3\end{bmatrix}$

Therefore, L_4 is not a linear transformation.

(b) For each L_i this is a linear transformation, find a matrix representing the linear transformation.

i. $L_1\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\right)=\begin{bmatrix}u_1-u_2\\u_3\end{bmatrix}$

Notice that $L_1\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}$, $L_1\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-1\\0\end{bmatrix}$, and $L_1\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}$

Then, taking these images of the columns of our matrix, the following matrix represents L_1: $A=\begin{bmatrix}1&-1&0\\0&0&1\end{bmatrix}$

ii. $L_2\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\right)=\begin{bmatrix}u_1-u_2\\0\end{bmatrix}$

Notice that $L_2\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}$, $L_2\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-1\\0\end{bmatrix}$, and $L_2\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\0\end{bmatrix}$

Then, taking these images of the columns of our matrix, the following matrix represents L_2: $A=\begin{bmatrix}1&-1&0\\0&0&0\end{bmatrix}$

(c) For each L_i this is a linear transformation, find the kernel and range of the linear transformation.

i. $L_1\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\right)=\begin{bmatrix}u_1-u_2\\u_3\end{bmatrix}$

Notice that to be in $ker\,L_1$, we must have $u_1=u_2$ and $u_3=0$. Therefore, $ker\,L_1=\left\{\begin{bmatrix}t\\t\\0\end{bmatrix}:t\in\mathbb{R}\right\}$.

To find the range, notice that if $\begin{bmatrix}a\\b\end{bmatrix}\in R^2$, then if we take $u_1=a$, $u_2=0$ and $u_3=b$, then $L_1\left(\begin{bmatrix}a\\0\\b\end{bmatrix}\right)=\begin{bmatrix}a\\b\end{bmatrix}$. Hence L_1 is onto. That is, $range\,L_1=R^2$.

ii. $L_2\left(\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\right)=\begin{bmatrix}u_1-u_2\\0\end{bmatrix}$

Notice that to be in $ker\,L_2$, we must have $u_1=u_2$, and u_3 can be anything. Therefore, $ker\,L_2=\left\{\begin{bmatrix}s\\s\\t\end{bmatrix}:s,t\in\mathbb{R}\right\}$.

Although this was not asked, one should notice that the kernel of L_1 has dimension 1, while the kernel of L_2 has dimension 2.

To find the range, notice that if $\begin{bmatrix} a \\ b \end{bmatrix} \in R^2$, then we must have $b = 0$. If we take $u_1 = a$, $u_2 = 0$ and $u_3 = b$, then $L_2\left(\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ 0 \end{bmatrix}$. Hence $range\, L_2 = \left\{\begin{bmatrix} a \\ 0 \end{bmatrix} : a \in \mathbb{R}\right\}$.

10. Suppose that L is a linear transformation with:

$$L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ L\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix},\ \text{and } L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

We will actually start by doing part (b):

(b) Find the standard matrix representing this linear transformation.

Notice that $L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$

Similarly, $L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = L\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$

From this, the matrix representing this linear transformation is: $A = \begin{bmatrix} -3 & 1 & 3 \\ 1 & -2 & 0 \\ -2 & 2 & 0 \end{bmatrix}$

(a) Find $L\left(\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}\right)$.

Using the matrix we found in part (b), $L\left(\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -3 & 1 & 3 \\ 1 & -2 & 0 \\ -2 & 2 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ 11 \\ -16 \end{bmatrix}$

(c) Determine whether or not L is one-to-one.

Notice that $det(A) = 0+0+6-12-0-0 = -6$. Since $det(A) \neq 0$, then $ker\, A = \{\vec{0}\}$. Hence L is one-to-one. (Although this was not asked, you should notice that since L is a transformation from R^3 to R^3, since L is one-to-one, it is also onto).

11. Let $L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1+u_2+u_3 \\ 0 \end{bmatrix}$

(a) Prove that L is a linear transformation.

Notice that $L(\vec{u}) + L(\vec{v}) = L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) + L\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} u_1+u_2+u_3 \\ 0 \end{bmatrix} + \begin{bmatrix} v_1+v_2+v_3 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} u_1+u_2+u_3+v_1+v_2+v_3 \\ 0+0 \end{bmatrix} = \begin{bmatrix} (u_1+v_1)+(u_2+v_2)+(u_3+v_3) \\ 0 \end{bmatrix}$

Similarly, $L(\vec{u}+\vec{v}) = L\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}\right) = \begin{bmatrix} (u_1+v_1)+(u_2+v_2)+(u_3+v_3) \\ 0 \end{bmatrix} = L(\vec{u}) + L(\vec{v})$. Hence property (a) holds.

Next, notice that $L(c \cdot \vec{u}) = L\left(\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}\right) = \begin{bmatrix} cu_1 + cu_2 + cu_3 \\ 0 \end{bmatrix} = c \cdot \begin{bmatrix} u_1 + u_2 + u_3 \\ 0 \end{bmatrix}$. Thus property (b) also holds. Therefore, L is a linear transformation.

(b) Show that L is not one-to-one and find a basis for $ker\, L$.

To be in $ker\, L$, we need $u_1 + u_2 + u_3 = 0$. That is, we must have $u_1 = -u_2 - u_3$. Let $u_2 = s$ and $u_3 = t$.

Then $ker\, L = \left\{ \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}$.

Therefore, the following set is a basis for $ker\, L$: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Note that since $ker, L \neq \{\vec{0}\}$, L is not one-to-one.

(c) Determine whether or not L is onto.

Notice that every element or the range of L is of the form $\begin{bmatrix} u_1 + u_2 + u_3 \\ 0 \end{bmatrix}$

From this, we see that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in the range of L. Hence L is not onto.

12. Let $L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + u_2 \\ u_3 - u_2 \end{bmatrix}$

(a) Prove that L is a linear transformation.

Notice that $L(\vec{u}) + L(\vec{v}) = L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) + L\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + u_2 \\ u_3 - u_2 \end{bmatrix} + \begin{bmatrix} v_1 + v_2 \\ v_3 - v_2 \end{bmatrix} = \begin{bmatrix} u_1 + u_2 + v_1 + v_2 \\ u_3 - u_2 + v_3 - v_2 \end{bmatrix} =$ $\begin{bmatrix} (u_1 + v_1) + (u_2 + v_2) \\ (u_3 + v_3) - (u_2 - v_2) \end{bmatrix}$

Similarly, $L(\vec{u} + \vec{v}) = L\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}\right) = \begin{bmatrix} (u_1 + v_1) + (u_2 + v_2) \\ (u_3 + v_3) - (u_2 + v_2) \end{bmatrix} = L(\vec{u}) + L(\vec{v})$. Hence property (a) holds.

Next, notice that $L(c \cdot \vec{u}) = L\left(\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}\right) = \begin{bmatrix} cu_1 + cu_2 \\ cu_3 - cu_2 \end{bmatrix} = c \cdot \begin{bmatrix} u_1 + u_2 \\ u_3 - u_2 \end{bmatrix}$. Thus property (b) also holds. Therefore, L is a linear transformation.

(b) Show that L is not one-to-one and find a basis for $ker\, L$.

To be in $ker\, L$, we need $u_1 + u_2 = 0$ and $u_3 - u_2 = 0$. That is, we must have $u_1 = -u_2$ and $u_3 = u_2$. Let $u_2 = t$.

Then $ker\, L = \left\{ \begin{bmatrix} -t \\ t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$.

Therefore, the following set is a basis for $ker\, L$: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Note that since $ker, L \neq \{\vec{0}\}$, L is not one-to-one.

(c) Determine whether or not L is onto.

Let $\begin{bmatrix} a \\ b \end{bmatrix} \in R^2$. Notice that if $u_1 = a$, $u_2 = 0$ and $u_3 = b$, then $L\left(\begin{bmatrix} a \\ 0 \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}$.

Hence L is onto.

13. Prove Theorem 6.1(a)

Theorem 6.1(a): Let $L : V \to W$ be a linear transformation. Then $L(\vec{0_V}) = \vec{0_W}$.

Proof: Recall that $\vec{0_v} + \vec{0_V} = \vec{0_V}$. Then $L(\vec{0_V}) = L(\vec{0_V} + \vec{0_V}) = L(\vec{0_V}) + L(\vec{0_V})$, where the last equality uses property (a) of linear transformations.

Then $L(\vec{0_V}) - L(\vec{0_V}) = L(\vec{0_V}) + L(\vec{0_V}) - L(\vec{0_V})$. So $\vec{0_W} = L(\vec{0_V})$. $\square$.

14. Prove Theorem 6.4(b)

Theorem 6.4(b): Let $L : V \to W$ be a linear transformation. L is one-to-one if and only if $ker\, L = \{\vec{0_V}\}$.

Proof:

First, suppose that L is one-to-one. Let $\vec{v} \in ker\, L$. Then $L(\vec{v}) = \vec{0_W}$. Recall from Theorem 6.1(a) above that $L(\vec{0_v}) = \vec{0_W}$. Therefore, since L is one-to-one, we must have $\vec{v} = \vec{0_v}$. Hence $ker\, L = \{\vec{0_V}\}$.

Next, suppose that $ker\, L = \{\vec{0_V}\}$. Suppose $L(\vec{v_1}) = L(\vec{v_2})$ for some $v_1, v_2 \in V$. Then, subtracting, $L(\vec{v_1}) - L(\vec{v_2}) = \vec{0_w}$. That is, using Theorem 6.1(b), $L(\vec{v_1} - \vec{v_2}) = \vec{0_w}$. Therefore, $\vec{v_1} - \vec{v_2}$ is an element of $ker\, L$. Hence, since $ker\, L = \{\vec{0_V}\}$, we must have $\vec{v_1} - \vec{v_2} = \vec{0_V}$, so $\vec{v_1} = \vec{v_2}$. Thus L is one-to-one. $\square$.

15. Given a linear transformation $L : V \to W$, suppose that $dim\, V = n$ and $dim\, W = m$ with $m > n$.

    (a) Could L be one-to-one? Justify your answer.

    Yes. Recall that L is one-to-one if and only if $ker\, L = \{\vec{0_V}\}$. Also, $dim(ker\, L) + dim(range\, L) = dim\, V = n$. Since $dim\, W = m > n$, it is possible to have $dim(range\, L) = n$. This would make $dim(ker\, L) = 0$, and hence L would be one-to-one.

    (b) Could L be onto? Justify your answer.

    No. Recall that L is onto if and only if $range\, L = W$. Then we must have $dim(range\, L) = dim\, W = m$. However, the maximum possible value for $dim(range\, L$ is n. Since $dim\, W = m > n$, it is not possible to have $dim(range\, L) = n$. Hence L cannot be onto.

    (c) Could L be invertible? Justify your answer.

    No. Recall that L is invertible if and only if L is both one-to-one and onto. We showed in part (b) above that L cannot be onto. Hence L is not invertible.

16. Consider the matrix: $\begin{bmatrix} 4 & -1 \\ 3 & 0 \end{bmatrix}$

    (a) Find the characteristic polynomial for each of this matrix.

    Notice that if $A = \begin{bmatrix} 4 & -1 \\ 3 & 0 \end{bmatrix}$, then $\lambda I_2 - A = \begin{bmatrix} \lambda - 4 & 1 \\ -3 & \lambda \end{bmatrix}$.

    Then $det(\lambda I_2 - A) = (\lambda - 4)\lambda + 3 = \lambda^2 - 4\lambda + 3$.

    Thus $p(\lambda) = \lambda^2 - 4\lambda + 3$ is the characteristic polynomial for this matrix.

    (b) Find the eigenvalues for this matrix.

    Notice that $p(\lambda) = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$. Therefore, the eigenvalues are the solutions to the characteristic equation $p(\lambda) = 0$. That is, solutions to $(\lambda - 1)(\lambda - 3) = 0$

    Hence the eigenvalues are $\lambda = 1$ and $\lambda = 3$.

(c) For each eigenvalue, find an associated eigenvector.

To find eigenvectors for each eigenvalue, we substitute each eigenvalue into the matrix $\lambda I_2 - A$ and then find the null space of the substituted version of the matrix.

If we take $\lambda = 1$, then $\lambda I_2 - A = \begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix}$

After adding an augmented row of zeros and then putting into reduced row echelon form, we obtain: $\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, we have $x_1 = \frac{1}{3}x_2$ and x_2 is free, so let $x_2 = t$.

Then the eigenvectors associated with this eigenvalue are all of the form: $\left\{\begin{bmatrix} \frac{1}{3}t \\ t \end{bmatrix}\right\}$. For example, if we set $t = 3$, one possible eigenvector is $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

If we take $\lambda = 3$, then $\lambda I_2 - A = \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix}$

After adding an augmented row of zeros and then putting into reduced row echelon form, we obtain: $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence, we have $x_1 = x_2$ and x_2 is free, so let $x_2 = t$.

Then the eigenvectors associated with this eigenvalue are all of the form: $\left\{\begin{bmatrix} t \\ t \end{bmatrix}\right\}$. For example, if we set $t = 1$, one possible eigenvector is $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

17. Consider the matrix: $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

(a) Find the characteristic polynomial for each of this matrix.

Notice that if $A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, then $\lambda I_3 - A = \begin{bmatrix} \lambda+1 & 0 & -1 \\ 0 & \lambda-3 & 1 \\ 0 & -1 & \lambda \end{bmatrix}$.

Then (using Maple to save a little work) $det(\lambda I_3 - A) = \lambda^3 - 2\lambda^2 - 2\lambda + 1$.

Thus $p(\lambda) = \lambda^3 - 2\lambda^2 - 2\lambda + 1$ is the characteristic polynomial for this matrix.

(b) Find the eigenvalues for this matrix.

Notice that $p(\lambda) = \lambda^3 - 2\lambda^2 - 2\lambda + 1 = (\lambda + 1)(\lambda^2 - 3\lambda + 1)$. Therefore, the eigenvalues are the solutions to the characteristic equation $p(\lambda) = 0$.

Hence the eigenvalues are $\lambda = -1$ and $\lambda = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$, where the last two come from applying to the quadratic portion of the factored form of $p(\lambda)$.

(c) For each eigenvalue, find an associated eigenvector.

To find eigenvectors for each eigenvalue, we substitute each eigenvalue into the matrix $\lambda I_3 - A$ and then find the null space of the substituted version of the matrix.

If we take $\lambda = -1$, then $\lambda I_3 - A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -4 & 1 \\ 0 & -1 & -1 \end{bmatrix}$

After adding an augmented row of zeros and then putting into reduced row echelon form, we obtain: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Hence, x_1 is free, $x_2 = 0$, and $x_3 = 0$. so let $x_1 = t$.

Then the eigenvectors associated with this eigenvalue are all of the form: $\left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \right\}$. For example, if we set $t = 1$, one possible eigenvector is $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

If we take $\lambda = \frac{3}{2} + \frac{\sqrt{5}}{2}$, then $\lambda I_3 - A = \begin{bmatrix} \frac{5}{2} + \frac{\sqrt{5}}{2} & 0 & -1 \\ 0 & -\frac{3}{2} + \frac{\sqrt{5}}{2} & 1 \\ 0 & -1 & \frac{3}{2} + \frac{\sqrt{5}}{2} \end{bmatrix}$

After adding an augmented row of zeros and then putting into reduced row echelon form, we obtain: $\begin{bmatrix} 1 & 0 & \frac{-2}{5+\sqrt{5}} & 0 \\ 0 & 1 & \frac{2}{\sqrt{5}+3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Hence, x_3 is free, $x_1 = \frac{2}{5+\sqrt{5}}$, and $x_2 = \frac{-2}{\sqrt{5}+3}$. so let $x_3 = t$.

Then the eigenvectors associated with this eigenvalue are all of the form: $\left\{ \begin{bmatrix} t \cdot \frac{2}{5+\sqrt{5}} \\ t \cdot \frac{-2}{\sqrt{5}+3} \\ t \end{bmatrix} \right\}$. For example, if we set $t = 1$, one possible eigenvector is $\vec{x} = \begin{bmatrix} \frac{2}{5+\sqrt{5}} \\ \frac{-2}{\sqrt{5}+3} \\ 1 \end{bmatrix}$

If we take $\lambda = \frac{3}{2} - \frac{\sqrt{5}}{2}$, then $\lambda I_3 - A = \begin{bmatrix} \frac{5}{2} - \frac{\sqrt{5}}{2} & 0 & -1 \\ 0 & -\frac{3}{2} - \frac{\sqrt{5}}{2} & 1 \\ 0 & -1 & \frac{3}{2} - \frac{\sqrt{5}}{2} \end{bmatrix}$

After adding an augmented row of zeros and then putting into reduced row echelon form, we obtain: $\begin{bmatrix} 1 & 0 & \frac{2}{-5+\sqrt{5}} & 0 \\ 0 & 1 & \frac{-2}{\sqrt{5}+3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Hence, x_3 is free, $x_1 = \frac{-2}{-5+\sqrt{5}}$, and $x_2 = \frac{2}{\sqrt{5}+3}$. so let $x_3 = t$.

Then the eigenvectors associated with this eigenvalue are all of the form: $\left\{ \begin{bmatrix} t \cdot \frac{-2}{-5+\sqrt{5}} \\ t \cdot \frac{2}{\sqrt{5}+3} \\ t \end{bmatrix} \right\}$. For example, if we set $t = 1$, one possible eigenvector is $\vec{x} = \begin{bmatrix} \frac{-2}{-5+\sqrt{5}} \\ \frac{2}{\sqrt{5}+3} \\ 1 \end{bmatrix}$