Math 327
Linear Independence

Recall: If $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ is a set of vectors in a vector space V, then the set of all vectors in V that are linear combinations of the vectors in S is denoted by **span S** or $span\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$. If $spanS = V$, then we say that S **spans** V or that V **is spanned by S**.

Definition: A set of vectors $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ drawn from the vector space V is said to be **linearly dependent** if there exist constants $a_1, a_2, \cdots, a_k$ **not all zero** so that $\sum_{j-1}^{k} a_j\vec{v_j} = a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_k\vec{v_k} = \vec{0}$.

Otherwise, we say that the set S is **linearly independent**. In this case, if $\sum_{j-1}^{k} a_j\vec{v_j} = a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_k\vec{v_k} = \vec{0}$, then we must have that $a_1 = a_2 = \cdots = a_k = 0$.

Example:

Theorem 4.5: Let $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ be a set of n vectors in R^n (or R_n). Let A be the matrix whose columns (rows) are the elements of S. Then S is linearly independent if and only if $det(A) \neq 0$.

Proof: We prove the case where $V = R^n$. The case for R_n is similar.

Suppose that S is linearly independent. Then the reduced row echelon form of A must be I_n and hence $det(A) \neq 0$ (another way to see this is to note this if $det(A) = 0$, then the related homogeneous system of equations must have a non-trivial solution, but that would give a non-trivial linear combination of the elements of A whose sum is equal to $\vec{0}$.)

Conversely, if $det(A) \neq 0$, then we know that the reduced row echelon form of A is equivalent to I_n. But then the related homogeneous system of equations has only the trivial solution. Hence there is no non-trivial linear combination of the columns of A that is equal to $\vec{0}$. Hence S is linearly independent. $\square$

Example:

Theorem 4.6: Let V be a vector space and suppose that S_1 and S_2 are subsets of V with $S_1 \subset S_2$. The the following both hold:

(a) If S_1 is linearly dependent then so is S_2.

(b) If S_2 is linearly independent then so is S_1.

Proof: Let $S_1 = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ and let $S_2 = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}, \vec{v_{k+1}}, \cdots, \vec{v_m}\}$.

(a) Suppose that S_1 is linearly dependent. Then there exist constants $a_1, a_2, \cdots, a_k$ **not all zero** so that $\sum_{j-1}^{k} a_j\vec{v_j} = \vec{0}$. Then, is we take $a_{k+1} = \cdots = a_m = 0$, then $\sum_{j-1}^{m} a_j\vec{v_j} = \vec{0}$, and at least one of the a_i for $i = 1, 2, \cdots k$ is nonzero. Hence S_2 is linearly dependent.

(b) Now suppose that S_2 is linearly independent. Then suppose $\sum_{j-1}^{k} a_j\vec{v_j} = \vec{0}$. If we take $a_{k+1} = \cdots = a_m = 0$, then, as before, $\sum_{j-1}^{m} a_j\vec{v_j} = \vec{0}$. Therefore, since S_2 is linearly independent, we must have $a_1 = a_2 = \cdots = a_m = 0$. But then a_i for $i = 1, 2, \cdots k$ in the original sum. Hence S_1 is also linearly independent. $\Box$

Notes:

1. The set $S = \{\vec{0}\}$ is linearly dependent since $a_1\vec{0} = \vec{0}$ for any $a_1 \in \mathbb{R}$.
2. From this and the theorem above, if $\vec{0} \in S$, then S is linearly dependent.
3. If $S = \{\vec{v}\}$ for any $\vec{v} \neq \vec{0}$, then S is linearly independent (since for all $a \in \mathbb{R}, a \neq 0$, we have $a\vec{v} \neq \vec{0}$.

Theorem 4.7: A set of non-zero vectors $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ in a vector space V are linearly dependent if and only if one of the vectors v_j, $j \geq 2$ is a linear combination of the set of preceding vectors: $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_{j-1}}\}$.

Proof: Suppose that $\vec{v_j} = a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_{j-1}\vec{v_{j-1}}$. Since $\vec{v_j} \neq \vec{0}$, at least one $a_i \neq 0$. Then $a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_{j-1}\vec{v_{j-1}} - \vec{v_j} = \vec{0}$. Consequently, the set $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_j}\}$ is linearly dependent. Thus, either $j = k$ and S is linearly dependent, or we may apply Theorem 4.6(a) to $S_1 = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_j}\}$ and conclude that S is also linearly dependent.

Conversely, suppose that S is linearly dependent. Then $a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_k\vec{v_k} = \vec{0}$ with at least one $a_i \neq 0$. Let j be the largest index such that $a_j \neq 0$. Since all vectors in S are non-zero, we must have $j \geq 2$. Therefore, $a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_j\vec{v_j} = \vec{0}$, so $a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_{j-1}\vec{v_{j-1}} = -a_j\vec{v_j}$. Since $a_j \neq 0$, we may divide by $-a_j$ to obtain:

$v_j = -\frac{a_1}{a_j}\vec{v_1} - \frac{a_2}{a_j}\vec{v_2} + \cdots - \frac{a_{j-1}}{a_j}\vec{v_{j-1}}$. $\Box$

Now that we understand the concepts of span and linear independence, we can make the following definition:

Definition: A set of vectors $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ in a vector space V forms a **basis** for V if both of the following hold:

(a) $span S = V$ (that is, the vector space V is spanned by the set S)

(b) S is a linearly independent set.

Notes:

1. If $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ is a basis for a vector space V, then the v_i's must be distinct and non-zero.
2. One can extend the definition of a basis to allow an infinite set S (provided (a) and (b) are still satisfied).