Math 327
Matrix Multiplication Handout

Definitions:

- Given two n-dimensional vectors in $\mathbb{R}^n$, $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, the **dot product** (or **inner product**) of $\vec{a}$ and $\vec{b}$, $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n = \sum_{i=1}^{n} a_ib_i$

  Notice that the dot product of two vectors is always a scalar.

  Example: Let $\vec{a} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$.
  Then $\vec{a} \cdot \vec{b} = (-1)(2) + (3)(-1) + (4)(0) = -2 - 3 + 0 = -5$.

- Suppose that $A = [a_{ij}]$ is an $m \times p$ matrix and $B = [b_{ij}]$ is a $p \times n$ matrix. Then the **product** of A and B, written AB is the $m \times n$ matrix $C = [c_{ij}]$ defined by: $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$.

  Note: In order to multiply two matrices, the matrices must be of appropriate sizes relative to one another. More specifically, the number of columns in the first matrix must be that same as the number of rows in the second matrix.

  This definition may seem a bit strange, but here is an illustration of how it works:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & & & \vdots \\ & & \boxed{c_{ij}} & \\ \vdots & & & \vdots \\ c_{m1} & a_{c2} & \cdots & c_{mn} \end{bmatrix}$$

  Notice that $c_{ij} = [\text{row}_i(A)]^T \cdot [\text{column}_j(B)] = \sum_{k=i}^{p} a_{ik}b_{kj}$.

  Example 1: Let

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & 5 \\ -1 & 6 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 \\ -2 & 5 \\ 6 & -1 \end{bmatrix}$$

  .

  Find AB (Notice that the (2,1) entry of AB is found by computing $[\text{row}_2(A)]^T \cdot [\text{column}_1(B)]$)

  Notice that BA is undefined.

Example 2: Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$$

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Find $AB =$

Find $BA =$

Notice that $AB \neq BA$.

Representing Linear Systems Using Matrices:

Consider the linear system given by:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

If we define

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Then $A\vec{x} = \vec{b}$ represents the given system of linear equations.

- We call A the **coefficient matrix** of the linear system.
- We can also adjoin the column b to the matrix A to obtain the **augmented matrix** for the linear system, $[A|b]$

Example: Given the system:

$$\begin{aligned} x - 2y + 3z &= 0 \\ -3x + y - z &= -2 \\ 2x + y - 3z &= 1 \end{aligned}$$

The matrix representation is:

While the augmented matrix is: