Math 327
Algebraic Properties of Matrix Operations

A. Properties of Matrix Addition:

Theorem 1.1 Let A, B, and C be $m \times n$ matrices. Then the following properties hold:

a) $A + B = B + A$ (commutativity of matrix addition)

b) $A + (B + C) = (A + B) + C$ (associativity of matrix addition)

c) There is a unique matrix O such that $A + O = A$ for any $m \times n$ matrix A. The matrix O is called the **zero matrix** and serves as the *additive identity* for the set of $m \times n$ matrices. Note that O is the $m \times n$ matrix with each entry equal to zero.

d) For each $m \times n$ matrix A there is a unique $m \times n$ matrix D such that $A + D = O$. We usually denote D as $-A$ so we have $A + (-A) = O$. We call $-A$ and *additive inverse* of the matrix A. Note that $-A = (-1)A$.

Proof: We will prove part (b). Parts (a) and (c) are proved in your textbook, and part (d) can be proved using similar methods.
Let $A = [a_{ij}]$, $B = [b_{ij}]$, and $C = [c_{ij}]$ be $m \times n$ matrices. Let $A+(B+C) = D = [d_{ij}]$ and $(A+B)+C = E = [e_{ij}]$. Notice that for each (i, j), $d_{ij} = a_{ij} + (b_{ij} + c_{ij}) = (a_{ij} + b_{ij}) + c_{ij} = e_{ij}$ (using the associativity of addition of real numbers). Thus $D = E$. $\square$

Example: Suppose $A = \begin{bmatrix} 1 & -2 \\ x & \frac{1}{3} \end{bmatrix}$ and $B = \begin{bmatrix} y & z \\ \frac{1}{4} & w \end{bmatrix}$ with $A + B = O$. Find w, x, y, z.

B. Properties of Matrix Multiplication:

Theorem 1.2 Let A, B, and C be matrices of appropriate sizes. Then the following properties hold:

a) $A(BC) = (AB)C$ (associativity of matrix multipliction)

b) $(A + B)C = AC + BC$ (the right distributive property)

c) $C(A + B) = CA + CB$ (the left distributive property)

Proof: We will prove part (a). Parts (b) and (c) are left as homework exercises.

Let $A = [a_{ij}]$ be an $m \times n$ matrix, $B = [b_{ij}]$ an $n \times p$ matrix, and $C = [c_{ij}]$ be a $p \times q$ matrix. Suppose $AB = D = [d_{ij}]$ and $BC = E = [e_{ij}]$. Further suppose that $(AB)C = DC = F = [f_{ij}]$ and $A(BC) = AE = G = [g_{ij}]$.

Notice that for each (i, j), $d_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. Similarly, $e_{ij} = \sum_{l=1}^{p} b_{il} c_{lj}$.

Then $f_{ij} = \sum_{l=1}^{p} d_{il} c_{lj} = \sum_{l=1}^{p} \left(\sum_{k=1}^{n} a_{ik} b_{kl} \right) c_{lj} = \sum_{l=1}^{p} (a_{i1} b_{1l} + a_{i2} b_{2l} + \cdots + a_{in} b_{nl})\, c_{lj} = a_{i1} \sum_{l=1}^{p} b_{1l} c_{lj} + a_{i2} \sum_{l=1}^{p} b_{2l} c_{lj} + \cdots + a_{in} \sum_{l=1}^{p} b_{nl} c_{lj} = \sum_{k=1}^{n} a_{ik} \left(\sum_{l=1}^{p} b_{kl} c_{lj} \right) = \sum_{k=1}^{n} a_{ik} e_{kj} = g_{ij}$. Hence $F = G$. $\square$

C. Properties of Scalar Multiplication

Theorem 1.3 Let r, s be real numbers and A and B matrices of appropriate sizes. Then the following properties hold:

a) $r(sA) = (rs)A$

b) $(r + s)A = rA + sA$

c) $r(A + B) = rA + rB$

d) $A(rB) = r(AB) = (rA)B$

Proof: We will prove part (a). Parts (b), (c) and (d) are left as homework exercises.

Let $A = [a_{ij}]$ and suppose $r, s \in \mathbb{R}$. Suppose $r(sA) = B = [b_{ij}]$ and $(rs)A = C = [c_{ij}]$. Then for each (i, j), $b_{ij} = r(s \cdot a_{ij}) = (rs)a_{ij} = c_{ij}$ (using the associativity of multiplication of real numbers). Thus $B = C$. □

Examples:

D. Properties of the Transpose

Theorem 1.4 Let r be a real number and A and B matrices of appropriate sizes. Then the following properties hold:

a) $\left(A^T\right)^T = A$

b) $(A + B)^T = A^T + B^T$

c) $(AB)^T = B^T A^T$

d) $(rA)^T = rA^T$

Proof: We will prove part (b). Part (c) is proved in your textbook, while Parts (a) and (d) are left as homework exercises.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$, and suppose $(A + B)^T = C = [c_{ij}]$, while $A^T + B^T = D = [d_{ij}]$.
Notice that, by definition of the transpose of a matrix, c_{ij} is the (j, i) entry of $A + B$, so $c_{ij} = a_{ji} + b_{ji}$. Moreover, a_{ji} is the (i, j) entry of A^T. Similarly, b_{ji} is the (i, j) entry of B^T. Hence $a_{ji} + b_{ji} = d_{ij}$. Thus $c_{ij} = d_{ij}$. Therefore $C = D$. □.

Examples: