Math 327

The Rank of a Matrix

Definition: Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ be an $m \times n$ matrix.

(a) The rows of A, considered as vectors in R_n, span a subspace of R_n called the **row space** of A. The dimension of this subspace is called the **row rank** of the matrix A.

(b) The columns of A, considered as vectors in R^m, span a subspace of R^m called the **column space** of A. The dimension of this subspace is called the **column rank** of the matrix A.

Theorem 4.17: If A and B are two $m \times n$ row (column) equivalent matrices, then the row (column) space of A and B are equal.

Proof Sketch:

Example: Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

1. Find a basis for the column space of A.

2. Find basis for the row space of A.

3. Find the column rank of A and the row rank of A.

Theorem 4.18: The row rank and the column rank of an $m \times n$ matrix $[a_{ij}]$ are equal.

Proof Sketch:

Definition: Since the row rank and column rank of a matrix always agree, we will call this number the **rank** of the matrix A.

Recall: The **nullity** of a matrix A is the dimension of the null space of the matrix.

Theorem 4.19: If A is an $m \times n$ matrix, then rank $A+$ nullity $A = n$.

Proof Sketch:

Theorem 4.20: If A is an $n \times n$ matrix, then rank $A = n$ if and only if A is row equivalent to I_n.

Proof:

Corollary 4.7: Let A be an $n \times n$ matrix. A is nonsingular if and only if rank $A = n$.

Corollary 4.8: Let A be an $n \times n$ matrix. Rank $A = n$ if and only if $det(A) \neq 0$.

Corollary 4.9: Let A be an $n \times n$ matrix. The system $A\vec{x} = \vec{0}$ has non-trivial solutions if and only if rank $A < n$.

Corollary 4.10: Let A be an $n \times n$ matrix. The system $A\vec{x} = \vec{b}$ has a unique solution for any $n \times 1$ vector $\vec{b}$ if and only if rank $A = n$.

Note: We now know that the following statements are all equivalent for an $n \times n$ matrix A:

- A is non-singular.
- $A\vec{x} = \vec{0}$ has only the trivial solution.
- A is row (column) equivalent to I_n.
- For every $\vec{b}$ in R^n, $A\vec{x} = \vec{b}$ has a unique solution.
- A is the product of elementary matrices.
- $det(A) \neq 0$
- The rank $A = n$
- The nullity of A is zero.
- The rows of A form a linearly independent set in R_n.
- The columns of A form a linearly independent set in R^n.