Math 327
Exam 5 - Version 2 **Name:**______________________________

Instructions: You will have 55 minutes to complete this exam. The credit given on each problem will be proportional to the amount of correct work shown. Answers without supporting work will receive little credit.

1. Consider the matrix $A = \begin{bmatrix} 4 & -2 & 1 & 3 \\ -3 & 2 & 0 & 1 \\ 1 & 0 & 1 & 4 \end{bmatrix}$. The reduced row echelon form of A is: $\begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & \frac{3}{2} & \frac{13}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   (a) (5 points) Find a basis for the column space of A.

   Since we are given the reduced row echelon form of the matrix A, the simplest way to find a basis for the column space of A is to take the original columns corresponding to the positions of the leading 1's in the the reduced row echelon form of A. From this, the following set is a basis for the column space of A:

   $$S = \left\{ \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

   (b) (5 points) Find a basis for the row space of A.

   Since we are given the reduced row echelon form of the matrix A, the simplest way to find a basis for the row space of A is to take the new rows corresponding to the positions of the leading 1's in the the reduced row echelon form of A (We could also put A^T into r.r.e.f. and use the position of the leading 1's, but this is much more work). From this, the following set is a basis for the row space of A:

   $$T = \left\{ \begin{bmatrix} 1 & 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \frac{3}{2} & \frac{13}{2} \end{bmatrix} \right\}$$

   (c) (5 points) Find a basis for the null space of A.

   To find the null space, we add an augmented column of zeros to the reduced row echelon form of the matrix A and then interpret the related equations. From this, we have $x_1 = -x_3 - 4x_4$, $x_2 = -\frac{3}{2}x_3 - \frac{13}{2}x_4$, with both x_3 and x_4 free variables. Then set $x_3 = s$ and $x_4 = t$.

   Then the elements in the null space of A are all of the form: $\left\{ \begin{bmatrix} -s - 4t \\ -\frac{3}{2}s - \frac{13}{2}t \\ s \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}$

   Therefore, (taking $s = 1$, $t = 0$, and then $s = 0$, $t = 1$ the following set if a basis for the null space of A: $S = \left\{ \begin{bmatrix} -1 \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -\frac{13}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$

   (d) (2 points) What is the rank of A?

   Recall that $rank\, A$ is equal to both the row rank and the column rank of A. Then $rank\, A = 2$.

2. (5 points each) For each of the following statements, indicate whether the statement is True or False. You do not need to justify your answers.

   (a) Let A be an $n \times n$ matrix. If $det(A) \neq 0$ then $rank(A) = n$.

   True

   Since $det(A) \neq 0$, A is non-singular, so A is row equivalent to I_n. Thus A has n leading 1's in its r.r.e.f. Hence A has rank n.

   (b) Let $L : V \to W$ be a linear transformation between two finite dimensional vector spaces. If V_1 is a subspace of V, then $L(V_1)$ is a subspace of W.

   True.

   See section 6.1 homework exercise #32 (which was assigned and collected).

   (c) Let $L : V \to W$ be a linear transformation between two finite dimensional vector spaces. If $S = \{v_1, v_2, \cdots, v_n\}$ is a basis for V, then $T = \{L(v_1), L(v_2), \cdots, L(v_n)\}$ is a basis for the range of L.

   False

   Theorem 6.2 tells us that T spans the range of L. However, T need not be linearly independent. In fact, if $dimV > dimW$ and L is onto, then T cannot be linearly independent.

   (d) Let $L : V \to W$ be a linear transformation between two finite dimensional vector spaces. If L is onto, then $dim\, W \leq dim\, V$.

   True.

   This is a consequence of Theorem 6.6, which tells us that $dim\, ker, L + dim\, range\, L = \dim V$. Then $dim\, range\, L \leq \dim V$. If L is onto, $W = range\, L$, so $dim\, W = dim\, range\, L \leq \dim V$.

   (e) Let A and B be $n \times n$ matrices. If A and B have the same characteristic polynomial, then they also have the same eigenvalues.

   True.

   See Theorem 7.1

3. Let $L : R_3 \to R_3$ be defined by $L\left(\left[\begin{array}{ccc} u_1 & u_2 & u_3 \end{array}\right]\right) = \left[\begin{array}{ccc} u_1+u_3 & u_2 & u_2 \end{array}\right]$.

(a) (8 points) Show that L is a linear transformation.

$L\left(\left[\begin{array}{ccc} u_1 & u_2 & u_3 \end{array}\right]\right) + L\left(\left[\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}\right]\right) = \left[\begin{array}{ccc} u_1+u_3 & u_2 & u_2 \end{array}\right] + \left[\begin{array}{ccc} v_1+v_3 & v_2 & v_2 \end{array}\right]$

$= \left[\begin{array}{ccc} u_1+u_3+v_1+v_3 & u_2+v_2 & u_2+v_2 \end{array}\right]$.

Similarly, $L\left(\left[\begin{array}{ccc} u_1 & u_2 & u_3 \end{array}\right] + \left[\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}\right]\right) = L\left(\left[\begin{array}{ccc} u_1+v_1 & u_2+v_2 & u_3+v_3 \end{array}\right]\right)$

$= \left[\begin{array}{ccc} (u_1+v_1)+(u_3+v_3) & u_2+v_2 & u_2+v_2 \end{array}\right] = \left[\begin{array}{ccc} u_1+u_3+v_1+v_3 & u_2+v_2 & u_2+v_2 \end{array}\right]$

$= L\left(\left[\begin{array}{ccc} u_1 & u_2 & u_3 \end{array}\right]\right) + L\left(\left[\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}\right]\right)$.

Also, $L\left(c \cdot \left[\begin{array}{ccc} u_1 & u_2 & u_3 \end{array}\right]\right) = L\left(\left[\begin{array}{ccc} cu_1 & cu_2 & cu_3 \end{array}\right]\right) = \left[\begin{array}{ccc} cu_1+cu_3 & cu_2 & cu_2 \end{array}\right]$

$= c\left[\begin{array}{ccc} u_1+u_3 & u_2 & u_2 \end{array}\right] = cL\left(\left[\begin{array}{ccc} u_1 & u_2 & u_3 \end{array}\right]\right)$

Since both properties of a linear transformation hold, L is a linear transformation.

(b) (6 points) Find a basis for the kernel of L.

Suppose that $L\left(\left[\begin{array}{ccc} u_1 & u_2 & u_3 \end{array}\right]\right) = \vec{0}$. Then $u_1 + u_3 = 0$, so $u_1 = -u_3$ and $u_2 = 0$.

Then $ker\, L = \{\left[\begin{array}{ccc} -t & 0 & t \end{array}\right] : t \in \mathbb{R}\}$.

Then a basis for $ker\, L$ is the following set: $\{\left[\begin{array}{ccc} -1 & 0 & 1 \end{array}\right]\}$.

(c) (6 points) Determine whether or not L is onto.

The simplest way to observe that L is not onto is to note that since $ker\, L$ has dimension 1, so $range\, L$ has dimension 2. Since the co-domain has dimension 3, we see that $range\, L$ is a proper subset of R_3. Hence L is not onto.

Alternatively, notice that the for each vector $\left[\begin{array}{ccc} w_1 & w_2 & w_3 \end{array}\right]$ in the range of L, $w_2 = w_3$, so $\left[\begin{array}{ccc} 1 & 0 & 1 \end{array}\right]$ is not in the range of L, so L cannot be onto.

4. (10 points) Prove the following: Let $L : V \to W$ be a linear transformation. If $L(v_1) = L(v_2)$, then $v_1 - v_2$ is in $ker\, L$.

   Let v_1, v_2 satisfy $L(v_1) = L(v_2)$. Then, using Theorem 6.1b, $L(v_1) - L(v_2) = L(v_1 - v_2) = \vec{0}$.

   Thus $v_1 - v_2 \in ker\, L$. $\square$

5. (7 points) Let $L : R^2 \to R^2$ be a linear trans. with $L\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. Find the standard matrix representing the linear transformation L.

   Notice that $L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \frac{1}{5}L\left(\begin{bmatrix} 0 \\ 5 \end{bmatrix}\right) = \frac{1}{5}L\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \frac{1}{5}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 6 \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}$.

   Then $L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 6 \end{bmatrix} - \begin{bmatrix} \frac{2}{5} \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{7}{5} \\ 4 \end{bmatrix}$.

   Then, taking these images of the columns of our matrix, the following matrix represents L: $A = \begin{bmatrix} -\frac{7}{5} & \frac{1}{5} \\ 4 & 1 \end{bmatrix}$

6. Given a linear transformation $L : R^5 \to R^6$.

   (a) (5 points) Could L be one-to-one? Justify your answer.

   Notice that the domain of V has dimension 5, while the co-domain has dimension 6. In order for L to be one-to-one, we the range must have dimension 5, which would make the kernel trivial (dimension zero). Since the dimension of the co-domain is 6, it is possible for the range to have dimension 5. Hence L could be one-to-one. (Recall that a linear transformation is one-to-one if and only if the kernel is trivial).

   (b) (5 points) Could L be onto? Justify your answer.

   Notice that the domain of V has dimension 5, while the co-domain has dimension 6. Recall that $dim\, ker\, L + dim\, range\, L = dim\, R^5 = 5$. Therefore, the range of L has dimension at most 5. Since this is less than the dimension of the co-domain, L cannot be onto.

7. Let $A = \begin{bmatrix} 2 & -1 & 2 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

(a) (8 points) Find the characteristic polynomial of the matrix A.

Notice that $\lambda I_3 - A = \begin{bmatrix} \lambda - 2 & 1 & -2 \\ -3 & \lambda & -1 \\ 0 & -1 & \lambda \end{bmatrix}$.

Then $det(\lambda I_3 - A) = (\lambda - 2)\lambda^2 + 0 + (-6) - 0 - (-3\lambda) - (\lambda - 2) = \lambda^3 - 2\lambda^2 + 2\lambda - 4$.

Thus $p(\lambda) = \lambda^3 - 2\lambda^2 + 2\lambda - 4$ is the characteristic polynomial for this matrix.

(b) (8 points) Given that $\lambda = 2$ is an eigenvalue of A, find an eigenvector associated with this eigenvalue.

If we take $\lambda = 2$, then $\lambda I_3 - A = \begin{bmatrix} 0 & 1 & -2 \\ -3 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \rightarrow r_1 \leftrightarrow r_2 \begin{bmatrix} -3 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \overset{r_1 - 2r_3 \rightarrow r_1}{r_2 + r_3 \rightarrow r_3}$

$\begin{bmatrix} -3 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{3}r_1 \rightarrow r_1 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

After adding an augmented row of zeros we obtain: $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Hence, $x_1 = x_3$, $x_2 = 2x_3$, and x_3 is free. so let $x_3 = t$.

Then the eigenvectors associated with this eigenvalue are all of the form: $\left\{ \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} \right\}$.

For example, if we set $t = 1$, one possible eigenvector is $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$