Math 327
Subspaces

Definition: Let V be a vector space and W a non-empty subset of V. If W is a vector space with respect to the operations in V, then W is called a **subspace** of V.

Theorem 4.3 Let V be a vector space with operations $\oplus$ and $\odot$ and let W be a non-empty subset of V. Then W is a subspace of V if and only if the following conditions hold:
(a) If $\vec{u}$ and $\vec{v}$ are any elements in W, then $\vec{u} \oplus \vec{v}$ is in W.

(b) If $\vec{u}$ is any element in W and c is any real number, then $c \odot \vec{u}$ is in W (so we say that W is closed under the operation $\odot$).

Proof:

Note: We say that (a) and (b) above are **closure properties** for $\oplus$ and $\odot$ respectively.

Examples:

1. Let V be any vector space. Since V itself is a subset of V satisfying properties (a) and (b), then V is a subspace of itself.

2. Let V be any vector space and let $W = \{\vec{0}\}$. Since $\vec{0} \oplus \vec{0} = \vec{0}$ and $c \odot \vec{0} = \vec{0}$ for any real number c, then W is a subspace of V. We call $\{\vec{0}\}$ the **zero subspace** of V.

3. Last section, we showed that P, the set of all polynomials is a vector space and that P_n, the set of all polynomials of degree $\leq n$ for some fixed n is also a vector space (using the operations of function addition and function scalar multiplication). Since P_n is a subset of P, then P_n is a subspace of P for any n. Also, P_n is a subspace of P_{n+1} (in fact, P_k is a subspace of P_ℓ whenever $k \leq \ell$).

4. If we let T be the set of all polynomials of degree *exactly* 2, then T is *not* a subspace of P. The set T is a subset of P, but, as we showed before, T does not satisfy the closure property for function addition.

5. Consider the following subsets of $\mathbb{R}_3$:

   - $W_1 = \{\begin{bmatrix} a & b & c \end{bmatrix} : a \geq 0, b \geq 0, c \geq 0\}$
   - $W_2 = \{\begin{bmatrix} a & b & c \end{bmatrix} : a = 0\}$
   - $W_3 = \{\begin{bmatrix} a & b & c \end{bmatrix} : b = c\}$
   - $W_4 = \{\begin{bmatrix} a & b & c \end{bmatrix} : a - b = c\}$

   Which of these subsets are subspaces of $\mathbb{R}_3$?

Definition: Let $\vec{v_1}, \vec{v_2}, \cdots \vec{v_k}$ be vectors in a vector space V. A vector $\vec{v}$ is called a **linear combination** of $\vec{v_1}, \vec{v_2}, \cdots \vec{v_k}$ if $v = a_1\vec{v_1} + a_2\vec{v_2} + \cdots + a_k\vec{v_k} = \sum_{j=1}^{k} a_j\vec{v_j}$ for some real numbers $a_1, a_2, \cdots, a_k$.

Examples:

1. Recall $W_4 = \{\begin{bmatrix} a & b & c \end{bmatrix} : a-b=c\}$. Let $\vec{v_1} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$. Then for any $\vec{w} \in W$, $\vec{w} = a\vec{v_1} + b\vec{v_2}$, so every element of W_4 is a linear combination of $\vec{v_1}$ and $\vec{v_2}$.

2. Let $\vec{v_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Is $\begin{bmatrix} 8 \\ 11 \end{bmatrix}$ a linear combination of $\vec{v_1}$ and $\vec{v_2}$?

3. Let $\vec{v_1} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{v_3} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$. Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ a linear combination of $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_3}$?

4. Consider the homogeneous system of equations represented by $A\vec{x} = \vec{0}$. Consider the set $W = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$. As we will show below, W is a subspace of $\mathbb{R}^n$. We call W the **solution space** of the homogeneous system of equations or the **null space** of the coefficient matrix A.