Similarly, the number of hours that machine Y will be used is 60, so we have

$$2x_1 + 2x_2 + 3x_3 = 60.$$

Mathematically, our problem is to find nonnegative values of x_1, x_2, and x_3 so that

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 80 \\ 2x_1 + 2x_2 + 3x_3 &= 60. \end{aligned}$$

This linear system has infinitely many solutions. Following the method of Example 4, we see that all solutions are given by

$$\begin{aligned} x_1 &= \frac{20 - x_3}{2} \\ x_2 &= 20 - x_3 \\ x_3 &= \text{any real number such that } 0 \le x_3 \le 20, \end{aligned}$$

since we must have $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$. When $x_3 = 10$, we have

$$x_1 = 5, \qquad x_2 = 10, \qquad x_3 = 10$$

while

$$x_1 = \tfrac{13}{2}, \qquad x_2 = 13, \qquad x_3 = 7$$

when $x_3 = 7$. The reader should observe that one solution is just as good as the other. There is no best solution unless additional information or restrictions are given. ■

As you have probably already observed, the method of elimination has been described, so far, in general terms. Thus we have not indicated any rules for selecting the unknowns to be eliminated. Before providing a very systematic description of the method of elimination, we introduce in the next section the notion of a matrix. This will greatly simplify our notational problems and will enable us to develop tools to solve many important applied problems.

Key Terms

Linear equation
Solution of a linear equation
Linear system
Unknowns
Inconsistent system
Consistent system
Homogeneous system
Trivial solution
Nontrivial solution
Equivalent systems
Unique solution
No solution
Infinitely many solutions
Manipulations on linear systems
Method of elimination

1.1 Exercises

In Exercises 1 through 14, solve each given linear system by the method of elimination.

1. $\begin{aligned} x + 2y &= 8 \\ 3x - 4y &= 4 \end{aligned}$

2. $\begin{aligned} 2x - 3y + 4z &= -12 \\ x - 2y + z &= -5 \\ 3x + y + 2z &= 1 \end{aligned}$

3. $\begin{aligned} 3x + 2y + z &= 2 \\ 4x + 2y + 2z &= 8 \\ x - y + z &= 4 \end{aligned}$

4. $\begin{aligned} x + y &= 5 \\ 3x + 3y &= 10 \end{aligned}$

5. $\begin{aligned} 2x + 4y + 6z &= -12 \\ 2x - 3y - 4z &= 15 \\ 3x + 4y + 5z &= -8 \end{aligned}$

6. $\begin{aligned} x + y - 2z &= 5 \\ 2x + 3y + 4z &= 2 \end{aligned}$

7. $\begin{aligned} x + 4y - z &= 12 \\ 3x + 8y - 2z &= 4 \end{aligned}$

8. $\begin{aligned} 3x + 4y - z &= 8 \\ 6x + 8y - 2z &= 3 \end{aligned}$

9. $\begin{aligned} x + y + 3z &= 12 \\ 2x + 2y + 6z &= 6 \end{aligned}$

10. $\begin{aligned} x + y &= 1 \\ 2x - y &= 5 \\ 3x + 4y &= 2 \end{aligned}$

11. $\begin{aligned} 2x + 3y &= 13 \\ x - 2y &= 3 \\ 5x + 2y &= 27 \end{aligned}$

12. $\begin{aligned} x - 5y &= 6 \\ 3x + 2y &= 1 \\ 5x + 2y &= 1 \end{aligned}$

13. $\begin{aligned} x + 3y &= -4 \\ 2x + 5y &= -8 \\ x + 3y &= -5 \end{aligned}$

14. $\begin{aligned} 2x + 3y - z &= 6 \\ 2x - y + 2z &= -8 \\ 3x - y + z &= -7 \end{aligned}$

15. Given the linear system

$$\begin{aligned} 2x - y &= 5 \\ 4x - 2y &= t, \end{aligned}$$

(a) Determine a particular value of t so that the system is consistent.

(b) Determine a particular value of t so that the system is inconsistent.

(c) How many different values of t can be selected in part (b)?

16. Given the linear system

$$\begin{aligned} 3x + 4y &= s \\ 6x + 8y &= t, \end{aligned}$$

(a) Determine particular values for s and t so that the system is consistent.

(b) Determine particular values for s and t so that the system is inconsistent.

(c) What relationship between the values of s and t will guarantee that the system is consistent?

17. Given the linear system

$$\begin{aligned} x + \quad\quad 2y &= 10 \\ 3x + (6 + t)y &= 30, \end{aligned}$$

(a) Determine a particular value of t so that the system has infinitely many solutions.

(b) Determine a particular value of t so that the system has a unique solution.

(c) How many different values of t can be selected in part (b)?

18. Is every homogeneous linear system always consistent? Explain.

19. Given the linear system

$$\begin{aligned} 2x + 3y - z &= 0 \\ x - 4y + 5z &= 0, \end{aligned}$$

(a) Verify that $x_1 = 1$, $y_1 = -1$, $z_1 = -1$ is a solution.

(b) Verify that $x_2 = -2$, $y_2 = 2$, $z_2 = 2$ is a solution.

(c) Is $x = x_1 + x_2 = -1$, $y = y_1 + y_2 = 1$, and $z = z_1 + z_2 = 1$ a solution to the linear system?

(d) Is $3x$, $3y$, $3z$, where x, y, and z are as in part (c), a solution to the linear system?

20. Without using the method of elimination, solve the linear system

$$\begin{aligned} 2x + y - 2z &= -5 \\ 3y + z &= 7 \\ z &= 4. \end{aligned}$$

21. Without using the method of elimination, solve the linear system

$$\begin{aligned} 4x \quad\quad\quad\quad &= 8 \\ -2x + 3y \quad\quad &= -1 \\ 3x + 5y - 2z &= 11. \end{aligned}$$

22. Is there a value of r so that $x = 1$, $y = 2$, $z = r$ is a solution to the following linear system? If there is, find it.

$$\begin{aligned} 2x + 3y - z &= 11 \\ x - y + 2z &= -7 \\ 4x + y - 2z &= 12. \end{aligned}$$

23. Is there a value of r so that $x = r$, $y = 2$, $z = 1$ is a solution to the following linear system? If there is, find it.

$$\begin{aligned} 3x \quad\quad - 2z &= 4 \\ x - 4y + z &= -5 \\ -2x + 3y + 2z &= 9. \end{aligned}$$

24. Show that the linear system obtained by interchanging two equations in (2) is equivalent to (2).

25. Show that the linear system obtained by adding a multiple of an equation in (2) to another equation is equivalent to (2).

26. Describe the number of points that simultaneously lie in each of the three planes shown in each part of Figure 1.2.

27. Describe the number of points that simultaneously lie in each of the three planes shown in each part of Figure 1.3.

28. Let C_1 and C_2 be circles in the plane. Describe the number of possible points of intersection of C_1 and C_2. Illustrate each case with a figure.

29. Let S_1 and S_2 be spheres in space. Describe the number of possible points of intersection of S_1 and S_2. Illustrate each case with a figure.

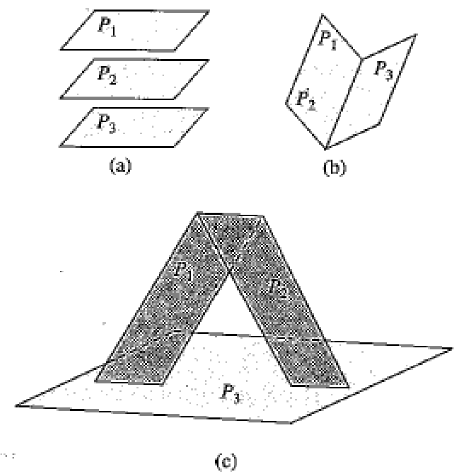

FIGURE 1.3

30. An oil refinery produces low-sulfur and high-sulfur fuel. Each ton of low-sulfur fuel requires 5 minutes in the blending plant and 4 minutes in the refining plant; each ton of high-sulfur fuel requires 4 minutes in the blending plant and 2 minutes in the refining plant. If the blending plant is available for 3 hours and the refining plant is available for 2 hours, how many tons of each type of fuel should be manufactured so that the plants are fully used?

31. A plastics manufacturer makes two types of plastic: regular and special. Each ton of regular plastic requires 2 hours in plant A and 5 hours in plant B; each ton of special plastic requires 2 hours in plant A and 3 hours in plant B. If plant A is available 8 hours per day and plant B is available 15 hours per day, how many tons of each type of plastic can be made daily so that the plants are fully used?

32. A dietician is preparing a meal consisting of foods A, B, and C. Each ounce of food A contains 2 units of protein, 3 units of fat, and 4 units of carbohydrate. Each ounce of food B contains 3 units of protein, 2 units of fat, and 1 unit of carbohydrate. Each ounce of food C contains 3 units of protein, 3 units of fat, and 2 units of carbohydrate. If the meal must provide exactly 25 units of protein, 24 units of fat, and 21 units of carbohydrate, how many ounces of each type of food should be used?

33. A manufacturer makes 2-minute, 6-minute, and 9-minute film developers. Each ton of 2-minute developer requires 6 minutes in plant A and 24 minutes in plant B. Each ton of 6-minute developer requires 12 minutes in plant A and 12 minutes in plant B. Each ton of 9-minute developer requires 12 minutes in plant A and 12 minutes in plant B. If plant A is available 10 hours per day and plant B is available 16 hours per day, how many tons of each type of developer can be produced so that the plants are fully used?

34. Suppose that the three points $(1, -5)$, $(-1, 1)$, and $(2, 7)$ lie on the parabola $p(x) = ax^2 + bx + c$.

(a) Determine a linear system of three equations in three unknowns that must be solved to find a, b, and c.

(b) Solve the linear system obtained in part (a) for a, b, and c.

35. An inheritance of \$24,000 is to be divided among three trusts, with the second trust receiving twice as much as the first trust. The three trusts pay interest annually at the rates of 9%, 10%, and 6%, respectively, and return a total in interest of \$2210 at the end of the first year. How much was invested in each trust?

36. For the software you are using, determine the command that "automatically" solves a linear system of equations.

37. Use the command from Exercise 36 to solve Exercises 3 and 4, and compare the output with the results you obtained by the method of elimination.

38. Solve the linear system

$$\begin{aligned} x + \tfrac{1}{2}y + \tfrac{1}{3}z &= 1 \\ \tfrac{1}{2}x + \tfrac{1}{3}y + \tfrac{1}{4}z &= \tfrac{11}{18} \\ \tfrac{1}{3}x + \tfrac{1}{4}y + \tfrac{1}{5}z &= \tfrac{9}{20} \end{aligned}$$

by using your software. Compare the computed solution with the exact solution $x = \frac{1}{2}$, $y = \frac{1}{3}$, $z = 1$.

39. If your software includes access to a computer algebra system (CAS), use it as follows:

(a) For the linear system in Exercise 38, replace the fraction $\frac{1}{2}$ with its decimal equivalent 0.5. Enter this system into your software and use the appropriate CAS commands to solve the system. Compare the solution with that obtained in Exercise 38.

(b) In some CAS environments you can select the number of digits to be used in the calculations. Perform part (a) with digit choices 2, 4, and 6 to see what influence such selections have on the computed solution.

40. If your software includes access to a CAS and you can select the number of digits used in calculations, do the following: Enter the linear system

$$\begin{aligned} 0.71x + 0.21y &= 0.92 \\ 0.23x + 0.58y &= 0.81 \end{aligned}$$

into the program. Have the software solve the system with digit choices 2, 5, 7, and 12. Briefly discuss any variations in the solutions generated.