Math 327
Vector Spaces

Definition: A **real vector space** is a set V of elements on which we have two operations $\oplus$ and $\odot$ defined with the following properties:

(a) If $\vec{u}$ and $\vec{v}$ are any elements in V, then $\vec{u} \oplus \vec{v}$ is in V (we say that V is **closed** under the operation $\oplus$).

1. $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$ for all $\vec{u}, \vec{v} \in V$.
2. $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$ for all $\vec{u}, \vec{v}, \vec{w} \in V$.
3. There exists an element $\vec{0}$ in V such that $\vec{u} \oplus \vec{0} = \vec{0} \oplus \vec{u} = \vec{u}$ for any $\vec{u} \in V$.
4. For each $\vec{u} \in V$, there exists an element $-\vec{u} \in V$ such that $\vec{u} \oplus -\vec{u} = -\vec{u} \oplus \vec{u} = \vec{0}$.

(b) If $\vec{u}$ is any element in V and c is any real number, then $c \odot \vec{u}$ is in V (so we say that V is closed under the operation $\odot$).

5. $c \odot (\vec{u} \oplus \vec{v}) = c \odot \vec{u} \oplus c \odot \vec{v}$ for any $\vec{u}, \vec{v} \in V$ and any $c \in \mathbb{R}$.
6. $(c + d) \odot \vec{u} = c \odot \vec{u} \oplus d \odot \vec{u}$ for any $\vec{u} \in V$ and any $c, d \in \mathbb{R}$.
7. $c \odot (d \odot \vec{u}) = (cd) \odot \vec{u}$ or any $\vec{u} \in V$ and any $c, d \in \mathbb{R}$.
8. $1 \odot \vec{u} = \vec{u}$ for any $\vec{u} \in V$.

Note:

- The elements of V are called **vectors** and the elements of the set of real numbers $\mathbb{R}$ are called **scalars**.
- The operation $\oplus$ is called **vector addition** and the operation $\odot$ is called **scalar multiplication**.
- The vector $\vec{0}$ from property 3 is called a **zero vector**.
- The vector $-\vec{u}$ from property 4 is called a **negative of** $\vec{u}$.

Claim: The vector $\vec{0}$ is unique, and for a given vector $\vec{u}$, its negative $-\vec{u}$ is also unique.

We will prove this claim later...

Note: If we replace $\mathbb{R}$ with $\mathbb{C}$, the complex numbers, in the definition above, then we get a **complex vector space**. In fact, we can replace $\mathbb{R}$ with any field $\mathbb{F}$.

To specify a vector space, one must provide a set of vectors V, a set of scalars (for us, this will almost always be $\mathbb{R}$), and two operations: an "addition" and a "scalar multiplication" satisfying all of the properties in the definition above. The vectors in such an abstract vector space need not be thought of as "directed line segments". Anything that satisfies the definition will do.

Example 1: Let $\mathbb{R}^n$ be the set of all $n \times 1$ matrices with real number entries: $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$. Let $\oplus$ be matrix addition, and let $\odot$ be multiplying a matrix by a real number scalar.

The properties we proved in Section 1.4 show that $\mathbb{R}^n$ with these operations and the set $\mathbb{R}$ of scalars is a vector space.

Note: The special case $n = 1$ in the example above shows that $\mathbb{R}$ is itself a real vector space with scalar set $\mathbb{R}$ and with operations $+$ and $\cdot$ which in this case correspond to the usual real number addition and multiplication operations.

Example 2: The set of all $m \times n$ matrices with scalar set $\mathbb{R}$, matrix addition as $\oplus$ and matrix scalar multiplication as $\odot$ is a vector space, again due to the properties we proved in Section 1.4. We will denote this vector space by M_{mn}.

Example 3: Recall that a **polynomial** in the variable t is a function that can be expressed in the form: $p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots a_1 t + a_0$, where $a_0, a_1, \cdots, a_n$ are real numbers, n, a non-negative integer, and with $a_n \neq 0$. We say that the polynomial $p(t)$ has **degree** n. The zero polynomial is the function $p(t) = 0$ and has no degree.

Let P_n be the set of all polynomials of degree $\leq n$ together with the zero polynomial. Given two polynomials $p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots a_1 t + a_0$ and $q(t) = b_n t^n + b_{n-1} t^{n-1} + \cdots b_1 t + b_0$, and a real number c, we define the following operations:

$p(t) \oplus q(t) = (a_n + b_n) t^n + (a_{n-1} + b_{n-1}) t^{n-1} + \cdots (a_1 + b_1) t + (a_0 + b_0)$

$c \odot p(t) = (c a_n) t^n + (c a_{n-1}) t^{n-1} + \cdots (c a_1) t + (c a_0)$

Claim: P_n with the operations described above is a real vector space.

Proof: