Math 229
Algebra Review
01/14/2014

A. Exponents

Definition: $a^n = a \cdot a \cdot a \cdot a \cdot \ \ldots \ \cdot a$
(a multiplied by itself n times)

Properties:

1. $a^0 = 1$
2. $a^{-n} = \frac{1}{a^n}$
3. $a^m \cdot a^n = a^{m+n}$
4. $(a^m)^n = a^{mn}$
5. $(ab)^n = a^n b^n$
6. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
7. $\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}}$
8. $\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$
9. $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

C. Algebraic Expressions
An **algebraic expression** is the result of applying mathematical operations to some collection of variables and real numbers.
• A **monomial** is an expression of the form ax^n, where n is a natural number and a is a real number.
• A **polynomial** is any expression of the form $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$. When applying addition, subtraction, multiplication, and division to a polynomial, we simplify by combining "like terms".
Example: $3x^7 - 4x^5 + 12x^2 - 7x + 22$
Special Product and Factoring Formulas:

1. $(x - y)(x + y) = x^2 - y^2$
   [Difference of Squares]
2. $(x \pm y)^2 = x^2 \pm 2xy + y^2$
   [Perfect Square]
3. $(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3$
   [Perfect Cube]
4. $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
   [Sum of Cubes]
5. $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
   [Difference of Cubes]
6. $x^2 + y^2$
   [Sum of Squares - Does not factor!]

B. Radicals

Definition: Suppose n is a positive integer and a is a real number. Then we define the ***n*th root of *a***, denoted by $\sqrt[n]{a}$ as follows:
• If $a = 0$, then $\sqrt[n]{a} = 0$.
• If $a > 0$ then $\sqrt[n]{a}$ is the *positive* real number b such that $b^n = a$.
• If $a < 0$ and n is **odd**, then $\sqrt[n]{a}$ is the *negative* real number b such that $b^n = a$.
• If $a < 0$ and n is **even**, then $\sqrt[n]{a}$ is not a real number, since there is no real number b such that $b^n = a$.
Examples:
(a) $\sqrt[2]{9} = \sqrt{9} = 3$ since $3 \cdot 3 = 9$.
(b) $\sqrt[3]{-8} = -2$ since $(-2) \cdot (-2) \cdot (-2) = -8$.
(c) $\sqrt{-16}$ is not a real number. (notice that $4 \cdot 4 = 16$, and $(-4) \cdot (-4) = 16$)
Properties:

1. $(\sqrt[n]{a})^n = a$ if $\sqrt[n]{a}$ is a real number.
2. $\sqrt[n]{a^n} = a$ if $a \geq 0$.
3. $\sqrt[n]{a^n} = a$ if $a < 0$ and n is odd.
4. $\sqrt[n]{a^n} = |a|$ if $a < 0$ and n is even.
5. $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$ provided both exist.
6. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ provided both exist.
7. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$ provided both exist.

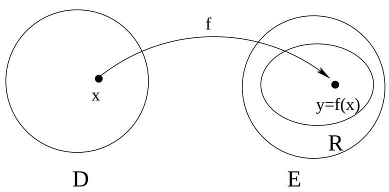
Warning!!
(a) In general, $\sqrt{a^2 + b^2} \neq a + b$
(b) Also, in general, $\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$

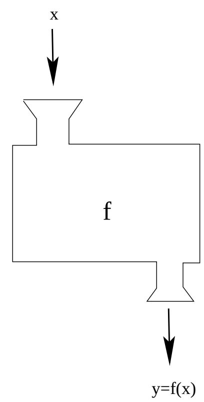
D. Algebraic Fractions
• An **algebraic fraction** is the quotient of two polynomials.
• To simplify algebraic fractions, we "factor and reduce".
• To change from division to multiplication, we multiply by the reciprocal.
• When adding or subtracting fractions, we need to find a *common denominator*

Examples:
(a) $\dfrac{x-3}{x^2-1} \div \dfrac{x^2-x-6}{x+1} = \dfrac{x-3}{x^2-1} \cdot \dfrac{x+1}{x^2-x-6}$
$= \dfrac{x-3}{(x+1)(x-1)} \cdot \dfrac{x+1}{(x-3)(x+2)} = \dfrac{1}{(x-1)(x+2)}$

(b) $\dfrac{2x}{x-2} - \dfrac{4}{+3} = \dfrac{2x(x+3)}{(x-2)(x+3)} - \dfrac{4(x-2)}{(x-2)(x+3)}$
$= \dfrac{2x^2+6x-4x+8}{(x-2)(x+3)} = \dfrac{2x^2+2x+8}{(x-2)(x+3)} = \dfrac{2(x^2+x+4)}{(x-2)(x+3)}$

Recall:
More formally, a function f from a domain set D to a set E is a correspondence that assigns to each element x of D exactly one element y of E. We call x the **argument** of f and y the **value** of f at x. The **range** of f is the subset R of E consisting of all y values that corresponding to an x in the domain D.





• To evaluate a function, we input an x-value and find the corresponding value by applying the "rule" for the function to that input.

• Sometimes we also want to work backwards, that is, given an **output**, we try to find the *input(s)* that lead to that particular output.

• To find the domain of a function, we carefully analyze the function "rule" and find any x values that do not have corresponding outputs. Two things we look for in particular are *division by zero* and *even roots of negative numbers.*

Example 1:
Suppose $f(x) = \frac{x+1}{x-1}$. Then:

$f(2) = \frac{2+1}{2-1} = \frac{3}{1} = 3$

$f(-1) = \frac{-1+1}{-1-1} = \frac{0}{-2} = 0$

$f(2a-1) = \frac{2a-1+1}{2a-1-1} = \frac{2a}{2a-2} = \frac{a}{a-1}$

If $f(x) = 2$, that what is x?

$\frac{x+1}{x-1} = 2$, so $x + 1 = 2(x-1) = 2x - 2$.

Then $x + 3 = 2x$, or $3 = x$. Check: $f(3) = \frac{3+1}{3-1} = \frac{4}{2} = 2$.

The domain of f is ? __________

Example 2:
Let $g(x) = \frac{\sqrt{3x-3}}{x^2+2x-3}$ Find:

• $g(4)$

• $g(1)$

• the domain of $g(x)$

II. Graphs of Functions

Definition:

The **graph** of a function is the set of all points $(x, f(x))$ (where x is in the domain D of f).

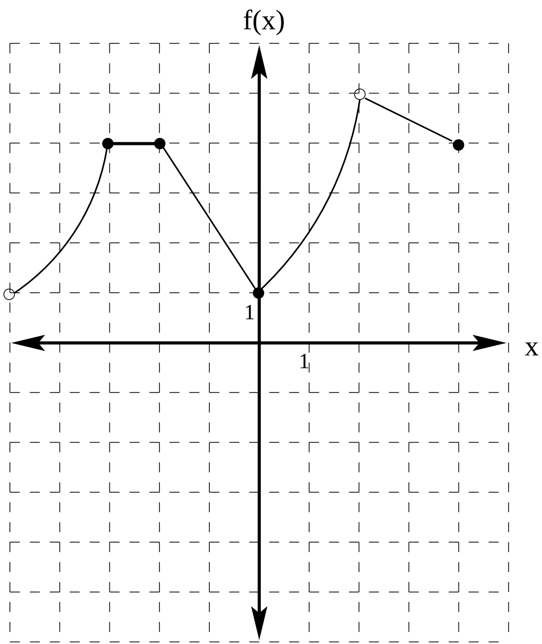
The Vertical Line Test:

A graph of points in the plane is the graph of a function if and only if every vertical line intersects the graph *at most* once.

Definitions:

A function is **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.
A function is **decreasing** on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.
A function is **constant** on an interval I if $f(x_1) = f(x_2)$ for all x_1, x_2 in I.

Example:



Find:
(a) $f(4)$

(b) x if $f(x) = 4$

(c) the domain of f

(d) the range of f

(e) the intervals where $f(x)$ is increasing