

## Differentiation

**A. The Formal Definition:** Given a function  $f(x)$ ,  $f'(x)$ , the **derivative** of  $f(x)$ , is a function that gives the slope of the tangent line to  $f(x)$  at any point  $x$  (provided such a slope exists). The function  $f'(x)$  is defined as

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

**Example:** Suppose  $f(x) = 2x^2 - 3x + 7$

$$\begin{aligned} \text{Then } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{2(x+h)^2 - 3(x+h) + 7 - (2x^2 - 3x + 7)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2(x^2 + 2xh + h^2) - 3x - 3h + 7 - (2x^2 - 3x + 7)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2x^2 + 4xh + 2h^2 - 3x - 3h + 7 - 2x^2 + 3x - 7}{h} = \lim_{h \rightarrow 0} \frac{4xh + 2h^2 - 3h}{h} = \lim_{h \rightarrow 0} 4x + 2h - 3 = 4x - 3 \end{aligned}$$

### B. Tangent Lines

The tangent line to a function  $f(x)$  when  $x = a$  is a line containing the point  $(a, f(a))$  with slope equal to the instantaneous rate of change of the function  $f$  when  $x = a$ . To find an equation for the tangent line of a function, we first find the point  $P = (a, f(a))$  by evaluating the function  $f$  when  $x = a$ . Then, we find the slope by finding the derivative function  $f'(x)$  and then evaluating the derivative function when  $x = a$ ,  $m = f'(a)$ . Finally, we apply the point/slope formula to the point  $P$  and the slope  $m$  to find the equation of the line.

**Example:** If  $f(x) = 2x^2 - 3x + 7$ , find the tangent line to  $f(x)$  when  $x = -1$ .

First notice that when  $x = -1$ ,  $f(-1) = 2(-1)^2 - 3(-1) + 7 = 2 + 3 + 7 = 12$ , so the point of tangency is  $P = (-1, 12)$ .

From above,  $f'(x) = 4x - 3$ , so  $m = f'(-1) = 4(-1) - 3 = -4 - 3 = -7$ .

Thus, by point/slope,  $y - 12 = -7(x + 1)$ , or  $y = -7x + 5$ .

### C. Basic Differentiation Formulas

1. Differentiating Power Functions:  $\frac{d}{dx}(x^r) = rx^{r-1}$  **Example:**  $\frac{d}{dx}(x^{\frac{2}{3}}) = \frac{2}{3}x^{-\frac{1}{3}}$

2. Differentiating a constant:  $\frac{d}{dx}c = 0$  for any constant  $c$ . **Example:**  $\frac{d}{dx}(12) = 0$

3. Constant Multiples:  $\frac{d}{dx}(cf(x)) = cf'(x)$  for any constant  $c$ . **Example:**  $\frac{d}{dx}(\frac{2}{3}x^3) = (\frac{2}{3})(3)x^2 = 2x^2$

4. Sums and Differences:  $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$ ,

and  $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$  for any functions  $f$  and  $g$ .

**Example:**  $\frac{d}{dx}(3x^2 - x^{\frac{3}{2}}) = 6x - \frac{3}{2}x^{\frac{1}{2}}$

5. Products:  $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

**Example:**  $\frac{d}{dx}((3x^3 - 4x + 7)(12x^4 - 13x^3 - 7x + 4))$   
 $= (9x^2 - 4)(12x^4 - 13x^3 - 7x + 4) + (3x^3 - 4x + 7)(48x^3 - 39x^2 - 7)$

6. Quotients:  $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

**Example:**  $\frac{d}{dx}\left(\frac{3x^3 - 4x + 7}{12x^4 - 13x^3 - 7x + 4}\right)$   
 $= \frac{(9x^2 - 4)(12x^4 - 13x^3 - 7x + 4) - (3x^3 - 4x + 7)(48x^3 - 39x^2 - 7)}{[12x^4 - 13x^3 - 7x + 4]^2}$