Math 229
Exam 3 Practice Problem Solutions

1. Solve the following equations (give exact answers whenever possible):

   (a) $e^{3x-2} = e^{4-5x}$
   By the one-to-one property of exponentials, $3x - 2 = 4 - 5x$.
   Then $8x = 6$, so $x = \frac{6}{8}$ or $x = \frac{3}{4}$

   (b) $9^{2x} = 27(3)^{2x+1}$
   Notice that all the terms can be written as powers of 3:
   $\left(3^2\right)^{2x} = 3^3(3)^{2x+1}$, or $3^{4x} = 3^{2x+4}$.
   Therefore, $4x = 2x + 4$, so $2x = 4$, or $x = 2$.

   (c) $\log_2(3x^2 - 3) = \log_2(x^2 + x)$
   Since the base on both logarithms are the same, by the one-to-one property, $3x^2 - 3 = x^2 + x$.
   Moving everything to one side, we have: $2x^2 - x - 3 = 0$, or $(2x - 3)(x + 1) = 0$
   Therefore, either $2x = 3$ or $x = -1$.
   That is, either $x = \frac{3}{2}$, or $x = -1$.
   However, notice that $x = -1$ does not check since you cannot take the logarithm of zero.
   Thus the only solution is $x = \frac{3}{2}$.

   (d) $\log_5(x^2 + 21) = 2$
   Rewriting in exponential form, we have $5^2 = x^2 + 21$, or $25 = x^2 + 21$.
   Therefore, $x^2 - 4 = 0$, or $(x + 2)(x - 2) = 0$. Thus either $x = 2$, or $x = -2$.
   Notice that both of these solutions check, since $x^2 + 21$ is positive when $x = \pm 2$.

   (e) $\log_2(2x) + \log_2(x - 3) = 3$
   Using the properties of logarithms to write this as a single logarithm, we have:
   $\log_2((2x)(x - 3)) = 3$, or $\log_2(2x^2 - 6x) = 3$.
   Changing this to exponential form, we have $2^3 = 2x^2 - 6x$.
   Moving everything to one side gives: $2x^2 - 6x - 8 = 0$, or $(2x - 8)(x + 1) = 0$. Thus either $2x = 8$, so $x = 4$, or $x = -1$.
   However, notice that $x = -1$ does not check since you cannot take the logarithm of a negative quantity.
   Thus the only solution is $x = 4$.

   (f) $e^{2x-1} = 3$
   Taking the natural log of both sides: $\ln\left(e^{2x-1}\right) = \ln 3$, or $2x - 1 = \ln 3$.
   Therefore, $2x = \ln 3 + 1$, so $x = \frac{\ln 3 + 1}{2}$

2. Determine whether the following are True or False:

   (a) $\ln\left(\dfrac{x^3}{(x+1)(x-1)}\right) = 3\ln x - \ln(x+1) + \ln(x-1)$
   False. Notice that $\ln\left(\dfrac{x^3}{(x+1)(x-1)}\right) = 3\ln x - \ln(x+1) - \ln(x-1)$.

   (b) $e^{\ln\left(x^2+1\right)} = x^2 + 1$
   True. This is the inverse function property of exponential and log functions.

   (c) $e^{x^2} \cdot e^{3x} = e^{3x^3}$
   False. In fact, $e^{x^2} \cdot e^{3x} = e^{x^2+3x}$. The exponents **add** rather than multiply here.

   (d) $\dfrac{\ln(4x)}{\ln(2x)} = \ln 2$
   False. This is not a legal simplification. For example, if $x = 1$, then $\dfrac{\ln(4x)}{\ln(2x)} = \dfrac{\ln(4)}{\ln(2)} = 2$, while $\ln 2 \approx .6931$.

   (e) $\ln\left(e^{x^2} - 4\right) = x^2 - \ln 4$
   False. The inverse function property does not apply here because of the subtracton operation. In fact, if $x = 2$, $\ln\left(e^{2^2} - 4\right) \approx 3.924$, while $2^2 - \ln 4 \approx 2.614$

3. Find the exact value of the following logarithmic expressions:

(a) $\log_2(32) = 5$ [since $2^5 = 32$] (c) $\log_5(1) = 0$ [since $5^0 = 1$]

(b) $\log_3\left(\frac{1}{27}\right) = -3$ [since $3^{-3} = \frac{1}{27}$] (d) $\log_4 32 = 2.5$ [since $2^5 = \left(4^{\frac{1}{2}}\right)^5 = 4^{\frac{5}{2}}$]

4. Use the laws of logarithms to simplify the expression: $\ln\left(\frac{x^2(x-1)^{\frac{5}{2}}}{(x-4)^3}\right)$

$$\ln\left(\frac{x^2(x-1)^{\frac{5}{2}}}{(x-4)^3}\right) = \ln x^2 + \ln(x-1)^{\frac{5}{2}} - \ln(x-4)^3 = 2\ln x + \frac{5}{2}\ln(x-1) - 3\ln(x-4)$$

5. Use the change of base formula to approximate the following:

(a) $\log_5 10$

By the change of base formula: $\log_b u = \dfrac{\log_a u}{\log_a b}$

Thus $\log_5 10 = \dfrac{\ln 10}{\ln 5} \approx 1.4307$

(b) $\log_9 12$

By the change of base formula: $\log_b u = \dfrac{\log_a u}{\log_a b}$

Thus $\log_9 12 = \dfrac{\ln 12}{\ln 9} \approx 1.1309$

(c) $\log_{15} 7$

By the change of base formula: $\log_b u = \dfrac{\log_a u}{\log_a b}$

Thus $\log_{15} 7 = \dfrac{\ln 7}{\ln 15} \approx 0.7186$

6. (a) Suppose you invest \$10,000 in a savings account that pays 3% annual interest compounded monthly. How much money will be in the account after 6 years?

$A = P(1+\frac{r}{n})^{nt} = 10,000(1+\frac{.03}{12})^{(12)(6)} = 10,000(1.0025)^{72} \approx \$11,969.48$

(b) How long would it take \$5,000 invested at 6% annual interest compounded continuously to triple?

The continuous interest formula is: $A = Pe^{rt}$. So we have $15,000 = 5,000e^{.06t}$, or $3 = e^{.06t}$, which makes sense since we want our initial investment to triple.
Taking the natural logarithm of both sides gives: $\ln 3 = \ln(e^{.06t}) = .06t$, so $t = \frac{\ln 3}{.06} \approx 18.31$ years.

(c) Find the interest rate needed for an investment of \$2,000 to double in 6 years if the interest is comounded quarterly.

Using the compound interest formula $A = P(1+\frac{r}{n})^{nt}$, we have $4,000 = 2,000(1+\frac{r}{4})^{(4)(6)}$, or $2 = (1+\frac{r}{4})^{24}$. Taking the natural log of both sides, $\ln 2 = \ln(1+\frac{r}{4})^{24} = 24\ln(1+\frac{r}{4})$, so $\frac{\ln 2}{24} = \ln(1+\frac{r}{4})$. Exponentiating both sides, we than have $e^{\frac{\ln 2}{24}} = e^{\ln(1+\frac{r}{4})} = 1+\frac{r}{4}$.

Hence $e^{\frac{\ln 2}{24}} - 1 = \frac{r}{4}$, therefore $4(e^{\frac{\ln 2}{24}} - 1) = r$, so $r \approx .1172$, or %11.72

7. Suppose that a culture of bacteria that initially has 500 cells grows to 10,000 cells in 12 hours.

(a) Find a function $f(t)$ that gives the number of cells in the culture as a function of time (in hours), assuming that this population grows continuously and exponentially.

Using the continuous exponential growth equation, $A = Pe^{rt}$, we see that $A = 10,000$, $p = 500$, and $t = 12$, or $10,000 = 500e^{12r}$. Therefore, $20 = e^{12r}$, so $\ln(20) = 12r$, and hence $r \approx .2496$.
Therefore, our function modeling the growth of this bacterial culture is: $f(t) = 500e^{.2496t}$.

(b) How long will it take for the culture to reach 1,000,000 cells?
Using the function $f(t) = 500e^{.2496t}$ found above, we solve $1,000,000 = 500e^{.2496t}$ for t.
Then $2000 = e^{.2496t}$, so $\ln(2000) = .2496t$, so $t = \frac{\ln(2000)}{.2496} \approx 30.45$ hours

8. Compute the derivatives of the following functions. You do not need to simplify your answers.

(a) $f(x) = e^{3x^2}$

$f'(x) = e^{3x^2} \cdot 6x = 6xe^{3x^2}$

(b) $g(x) = \ln(3x^2 - 4x + 6)$

$g'(x) = \frac{1}{3x^2-4x+6} \cdot (6x - 4) = \frac{6x-4}{3x^2-4x+6}$

(c) $h(x) = (x^2 + 1)e^{x^2+1}$

By the product rule, $h'(x) = 2xe^{x^2+1} + (x^2 + 1)e^{x^2+1} \cdot (2x) = 2xe^{x^2+1}(1 + x^2 + 1) = 2xe^{x^2+1}(x^2 + 2)$

(d) $k(x) = x^2 \ln(e^x + 1)$

Again using the product rule, $k'(x) = 2x \ln(e^x + 1) + x^2 \frac{1}{e^x+1} \cdot e^x = 2x \ln(e^x + 1) + \frac{x^2e^x}{e^x+1}$

(e) $f(x) = \ln\left(\frac{x^2}{(2x-1)^3}\right)$

Rather than just jumping in and differentiating, we can make our job a lot easier by simplifying $f(x)$ using properties of logarithms.

$f(x) = \ln(x^2) - \ln(2x - 1)^3 = 2 \ln x - 3 \ln(2x - 1)$

Then $f'(x) = 2 \cdot \frac{1}{x} - 3 \cdot \frac{2}{2x-1} = \frac{2}{x} - \frac{6}{2x-1}$

(f) $g(x) = e^{e^{2x^3}}$

Using the chain rule for exponential functions two times, we see:
$g'(x) = e^{e^{2x^3}} \cdot e^{2x^3} \cdot 6x^2$

(g) $h(x) = \ln(x^2 + 1)e^{x^3}$

By the product rule: $h'(x) = \frac{2x}{x^2+1}e^{x^3} + \ln(x^2 + 1)(3x^2)e^{x^3}$.

9. Find the tangent line to $f(x) = x \ln(2x)$ when $x = \frac{1}{2}$

First notice that $f(\frac{1}{2}) = \frac{1}{2} \ln(2\frac{1}{2}) = \frac{1}{2} \ln(1) = \left(\frac{1}{2}\right)(0) = 0$, so a point on our line is $P = (\frac{1}{2}, 0)$.

Next, to find the slope of the tangent line, we differentiate using the product rule:
$f'(x) = (1) \ln(2x) + x\frac{1}{2x} \cdot 2 = \ln(2x) + \frac{2x}{2x} = \ln(2x) + 1$.
Then $f'(\frac{1}{2}) = \ln(2 \cdot \frac{1}{2}) + 1 = \ln(1) + 1 = 0 + 1 = 1$, so $m = 1$.

Finally, using the point slope equation, $y - y_1 = m(x - x_1)$, we have $y - 0 = 1(x - \frac{1}{2})$, or $y = x - \frac{1}{2}$

10. Find the tangent line to $f(x) = 2xe^{2x-4}$ when $x = 2$
First, we find the y coordinate of the point of tangency by computing $f(2) = 2(2)e^{2(2)-4} = 4e^0 = 4$, so $P(2, 4)$ is the point of tangency.

Next, to find the slope of the tangent line, we compute $f'(x) = 2e^{2x-4} + (2x)(2)e^{2x-4}$, and evaluate when $x = 2$, $m = f'(2) = 2e^{2(2)-4} + (2)(2)(2)e^{2(2)-4} = 2e^0 + 8e^(0) = 2 + 8 = 10$.

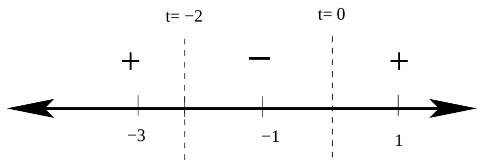
Finally, using point slope with $P(2, 4)$ and $m = 10$,
$y - 4 = 10(x - 2)$, or $y = 10x - 16$ is the equation of the tangent line when $x = 2$.

11. Determine the intervals where the function $g(t) = t^4 e^{2t}$ in increasing and the intervals where it is decreasing.

To find the intervals where $g(t)$ is increasing and those where $g(t)$ are decreasing, we analyze the first derivative of $g(t)$: $g'(t) = 4t^3e^{2t} + t^4(2)e^{2t} = 4t^3e^{2t} + 2t^4e^{2t} = 2t^3e^{2t}(2 + t)$.

To find the critical points, we solve the equation: $2t^3e^{2t}(2 + t) = 0$
We see that if $2t^3 = 0$, then $t = 0$, $e^{2t} = 0$ has no solution, and $2 + t = 0$ when $t = -2$, so we have two critical points, $t = 0$ and $t = -2$.

We use test points to determine the sign of the first derivative on each of the intervals between our critical points, as indicated in the following sign-testing diagram:



Therefore, $g(t)$ is increasing on the intervals $(-\infty, -2) \cup (0, \infty)$, and $g(t)$ is decreasing on the interval $(-2, 0)$.

12. Find the absolute extrema of $g(t) = t^2e^{2t}$ on the interval $[-2, 2]$.

Recall that absolute extrema occur either at critical points of our function, or at the endpoints of our interval.

We first find the critical points of g:
$g'(t) = 2te^{2t} + t^2e^{2t}(2) = 2te^{2t} + 2t^2e^{2t} = (2t)(e^{2t})(1 + t) = 0$

Since $g'(t)$ is continuous, the only critical points occur when the derivative is zero, or when $2t = 0$, $e^{2t} = 0$, or when $1 + t = 0$. Since e^{2t} is never zero, our solutions are $t = 0$, and $t = -1$.

Now, we can actually skip analyzing these critical points since classifying them is not necessary. So we just evaluate the original function on each critical point and on the two endpoints of the given interval and see what the highest and lowest values are:

$g(-2) = 4e^{-4} \approx .07326$
$g(2) = 4e^4 \approx 218.393$ – absolute maximum
$g(0) = 0e^0 = 0$ – absolute minimum
$g(-1) = 1e^{-2} \approx .1353$

13. Suppose $\int_0^2 f(x)\,dx = 4$ and $\int_0^2 g(x)\,dx = 2$. Find $\int_0^2 2f(x) - g(x)\,dx$

Using the properties of the definite integral:
$$\int_0^2 2f(x) - g(x)\,dx = \int_0^2 2f(x)\,dx - \int_0^2 g(x)\,dx = 2\int_0^2 f(x)\,dx - \int_0^2 g(x)\,dx = 2(4) - 2 = 6.$$

14. Evaluate the following integrals:

(a) $\int 6x^3 - 4x^{\frac{1}{2}}\,dx = 6\left(\frac{1}{4}\right)x^4 - 4\left(\frac{2}{3}\right)x^{\frac{3}{2}} + C = \frac{3}{2}x^2 - \frac{8}{3}x^{\frac{3}{2}} + C$

(b) $\int 5x^4 - x^{\frac{3}{2}}\,dx = x^5 - \frac{2}{5}x^{\frac{5}{2}} + C$

(c) $\int \frac{4x^3 - 3x^2 + 2x}{2x^2}\,dx = \int \frac{4x^3}{2x^2} - \frac{3x^2}{2x^2} + \frac{2x}{2x^2}\,dx = \int 2x - \frac{3}{2} + \frac{1}{x}\,dx = x^2 - \frac{3}{2}x\ln|x| + C$

(d) $\int_{-1}^{1} 3x^5 - 4x^3\ dx = 3\left(\frac{1}{6}\right)x^6 - 4\left(\frac{1}{4}\right)x^4\Big|_{-1}^{1} = \frac{1}{2}x^6 - x^4\Big|_{-1}^{1} = \left(\frac{1}{2} - 1\right) - \left(\frac{1}{2} - 1\right)$

(e) $\int_{0}^{3} 4x^2 - e^{3x}\ dx = \frac{4}{3}x^3 - \frac{1}{3}e^{3x}|_3^0 = \left(\frac{4}{3}(3)^3 - \frac{1}{3}e^{(3)(3)}\right) - \left(\frac{4}{3}(0)^3 - \frac{1}{3}e^{(0)(3)}\right)$

$= (36 - \frac{1}{3}e^9) - (0 - \frac{1}{3}) = 36 + \frac{1}{3} - \frac{1}{3}e^9 \approx -2664.67.$

(f) $\int_{0}^{4} e^{3x} + x^{-\frac{1}{2}}\ dx = \frac{1}{3}e^{3x} + 2x^{\frac{1}{2}}\Big|_0^4 = \left(\frac{1}{3}e^{12} + 2\sqrt{4}\right) - \left(\frac{1}{3}e^0 + 2\sqrt{0}\right)$

$= \frac{1}{3}e^{12} + 4 - \frac{1}{3} = \frac{1}{3}e^{12} + \frac{11}{3} \approx 54,255.263$

(g) $\int 5x(x^2+1)^8 dx$

Let $u = x^2 + 1$. Then $du = 2x\,dx$, or $\frac{1}{2}\,du = dx$, so we have $\int \frac{5}{2}u^8 du = \frac{5}{2}\frac{1}{9}u^9 + C = \frac{5}{18}u^9$.

Thus the indefinite integral is: $\frac{5}{18}(x^2+1)^9 + C$.

(h) $\int_0^1 x^2(2x^3+1)^2\ dx$

Let $u = 2x^3 + 1$. Then $du = 6x^2$, or $\frac{1}{6}du = dx$.

Notice $2(0)^3 + 1 = 1$, and $2(1)^3 + 1 = 3$

Then we have $\int_1^3 \frac{1}{6}u^2\ du = \frac{1}{18}u^3|_1^3 = \frac{1}{18}(3^3 - 1^3) = \frac{26}{18} = \frac{13}{9}$.

15. Assume f is continuous on $[-5, 3]$, $\int_{-5}^{-1} f(x)\ dx = -7$, $\int_{-1}^{3} f(x)\ dx = 4$, and $\int_{1}^{3} f(x)\ dx = 2$. Find:

(a) $\int_{3}^{-1} f(x)\ dx = -\int_{-1}^{3} f(x)\ dx = -4$

(b) $\int_{-5}^{1} f(x)\ dx = \int_{-5}^{-1} f(x)\ dx + \int_{-1}^{3} f(x)\ dx - \int_{1}^{3} f(x)\ dx = -7 + 4 - 2 = -5$

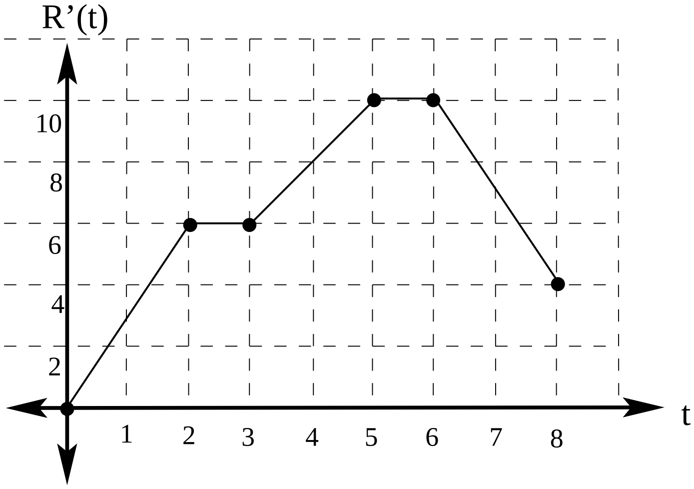
(c) $\int_{-5}^{3} f(x)\ dx = \int_{-5}^{-1} f(x)\ dx + \int_{-1}^{3} f(x)\ dx = -7 + 4 = -3$

(d) $\int_{-1}^{-1} f(x)\ dx = 0$

(e) Find the average value of f on $[-5, -1]$

$= \frac{1}{-1-(-5)}\int_{-5}^{-1} f(x)\ dx = \frac{1}{4}\cdot(-7) = -\frac{7}{4}$

16. Suppose marginal revenue, $R'(t)$ is given by the graph below, where t in in months and $R'(t)$ is in \$1000s of dollars per month. Find the total revenue from $t = 0$ to $t = 6$.



To find the total revenue from $t = 0$ to $t = 6$, we need to find the area under the marginal revenue function $R'(t)$ between $t = 0$ adn $t = 6$. We do not have a formula for $R'(t)$, but we do not need one, since we can actually use geometry to find this area (see the figure above).

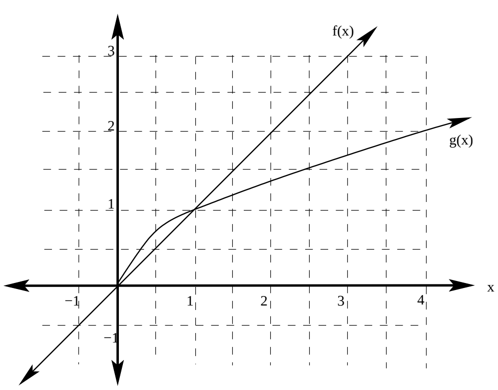
$A_1 = 3$ squares, $A_2 = 12$ squares, $A_3 = 2$ squares, and $A_4 = 2$ squares.

Since the total area under the curve from $t = 0$ to $t = 6$ is 19 squares, and each square corresponds to \$2000 of revenue, the total revenue for this time period is \$38,000.

17. Find the average value of $f(x) = x^2 - \frac{1}{x^2}$ for $1 \le x \le 3$.

Recall that Average Value $= \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\ dx = \dfrac{1}{3-1}\int_1^3 x^2 - x^{-2}\ dx = \dfrac{1}{2}\left[\dfrac{1}{3}x^3 + x^{-1}\Big|_1^3\right] = \dfrac{1}{2}\left[\left(\dfrac{27}{3}+\dfrac{1}{3}\right) - \left(\dfrac{1}{3}+1\right)\right]$
$= \frac{1}{2}\left[\frac{28}{3} - \frac{4}{3}\right] = 4$

18. Find the area of the region enclosed by the graphs $f(x) = x$ and $g(x) = \sqrt{x}$.

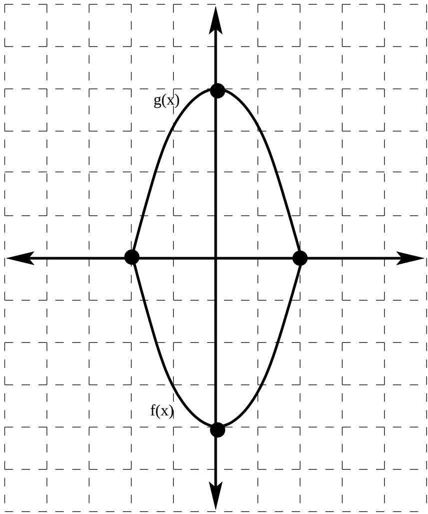


Using algebra, we see that if $f(x) = g(x)$, then $x = \sqrt{x}$, or, squaring both sides, $x^2 = x$. That is, $x^2 - x = 0$, or $x(x-1) = 0$. Therefore, $x = 0$ or $x = 1$. That is, these two functions cross when $x = 0$ and when $x = 1$. From the graph above, we see that there is a single region enclosed by these two functions, and that $g(x)$ is greater than $f(x)$ when x is between 0 and 1.

Therefore, $A = \displaystyle\int_0^1 g(x) - f(x)\ dx = \int_0^1 x^{\frac{1}{2}} - x\ dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2|_0^1$

$= \left(\frac{2}{3}(1)^{\frac{3}{2}} - \frac{1}{2}(1)^2\right) - \left(\frac{2}{3}(0)^{\frac{3}{2}} - \frac{1}{2}(0)^2\right) = \left(\frac{2}{3} - \frac{1}{2}\right) - (0-0) = \frac{1}{6}$

19. Find the area of the region enclosed by the graphs $f(x) = x^2 - 4$ and $g(x) = 4 - x^2$.



To find the region between the graphs, we need to know where the functions meet.
Notice if $f(x) = g(x)$, then $x^2 - 4 = 4 - x^2$, or $2x^2 = 8$
Thus $x^2 = 4$, or $x = \pm 2$. Since $f(2) = g(2) = 4 - 4 = 0$, and $f(-2) = g(-2) = 4 - 4 = 0$,
$f(x)$ and $g(x)$ both contain the points $(-2, 0)$ and $(2, 0)$.
Also notice that $g(x)$ is greater than $f(x)$ for $-2 < x < 2$.

Then the area enclosed by $f(x)$ and $g(x)$ is given by:

$$A = \int_{-2}^{2} g(x) - f(x)\ dx = \int_{-2}^{2} (4 - x^2) - (x^2 - 4)\ dx = \int_{-2}^{2} 8 - 2x^2\ dx = 8x - \frac{2}{3}x^3 \Big|_{-2}^{2}$$

$$= \left[16 - \tfrac{2}{3}(8)\right] - \left[-16 + \tfrac{2}{3}(8)\right] = 32 - \tfrac{32}{3} = \tfrac{64}{3} \approx 21.333$$