Math 229

Exponential Functions

A. Definition: An **exponential function** is a function of the form $f(x) = a^x$ for $0 < a < 1$ or $a > 1$. Note: a is called the **base** of the exponential function.

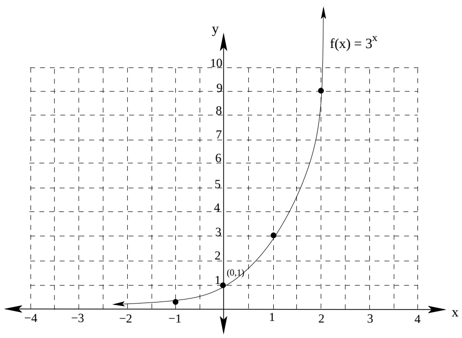
The reason we exclude 0 and 1 as bases for exponential function is because $0^x = 0$ for and x, and $1^x = 1$ for any x, so these are just constant functions.

Example 1: Let $f(x) = 3^x$. Then:
(a) $f(0) = 3^0 = 1$
(b) $f(2) = 3^2 = 9$
(c) $f(-3) = 3^{-3} = \frac{1}{27}$
(d) $f(\frac{2}{3}) = 3^{\frac{2}{3}} = \sqrt[3]{3^2} = \sqrt[3]{9} \approx 2.080084$

Graphs of exponential functions:

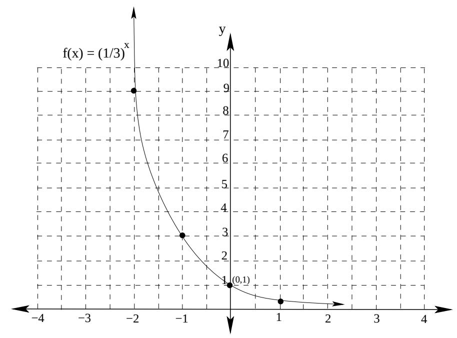
$f(x) = 3^x$

x	$f(x)$
-3	$\frac{1}{27}$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9
3	27



$f(x) = \left(\frac{1}{3}\right)^x$

x	$f(x)$
3	$\frac{1}{27}$
2	$\frac{1}{9}$
1	$\frac{1}{3}$
0	1
-1	3
-2	9
-3	27



Properties of Exponential Graphs:

1. Domain: $(-\infty, \infty)$
2. Range: $(0, \infty)$
3. y-intercept: $(0, 1)$, x intercept: none.
4. Continuous everywhere
5. Increasing if $b > 1$. Decreasing if $0 < b < 1$.

Solving Basic Exponential Equations:

Examples:
1. $4^{2x-3} = 4^{5-x}$
Since $f(x) = 4^x$ is a one-to-one function, we can conclude that:
$2x - 3 = 5 - x$, or $3x = 8$.
Hence $x = \frac{8}{3}$.

2. $2^{4x-7} = 8^{2x-5}$
Since $8 = 2^3$, we can rewrite 8^{2x-5} as $\left(2^3\right)^{2x-5} = 2^{3(2x-5)} = 2^{6x-15}$.
Then, as above, we know that $4x - 7 = 6x - 15$, or $8 = 2x$.
Hence $4 = x$.

The Compound Interest Formula: When a principal amount P in invested at interest rate r which is compounded n times per year and remains invested for t year, the amount A that results is given by the formula $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$

Examples:

1. Suppose you put \$1000 in an account that pays 6% interest compounded monthly. How much money will be in the account 3 years later?

   $P = 1000$, $r = 0.06$, $n = 12$, and $t = 3$, so $A = 1000\left(1 + \frac{.06}{12}\right)^{(12)(3)} = 1000\,(1.005)^{36} \approx \$1,196.68$

2. Now Suppose you put \$2000 in an account that pays 7% interest compounded daily. How much money will be in the account 5 years later?

   $P = 2000$, $r = 0.07$, $n = 365$, and $t = 5$, so $A = 2000\left(1 + \frac{.07}{365}\right)^{(365)(5)} \approx 2000\,(1.000191781)^{1825} \approx \$2838.04$

The Natural Exponential Function:

Definition: If we consider what happens to the base of our compound interest exponential term: $\left(1+\frac{1}{n}\right)$ as we compound more and more frequently

n	$\left(1+\frac{1}{n}\right)^n$
1	2.0
10	2.59374246
100	2.704813829
1,000	2.71692393
10,000	2.71814593
100,000	2.71826824
1,000,000	2.71828047

Therefore, we define $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. We say e is the **base of the natural exponential function**.

Continuously Compounded Interest Using this new base, we can measure the accumulation of interest that is compounded "instantaneously" rather than only n times a year. We do so using the formula: $A = Pe^{rt}$, where P, A, r, and t are exactly as above.

Example: Suppose you invest \$1000 at 6% interest compounded continuously for 3 years. Then at the end of the 3 years, you will have: $1000e^{0.06(3)} \approx \$1,197.22$
Notice that this is about 54 cents more that we had investing the same amount at the same interest rate but only compounded monthly.

Example: Suppose the population of a bacterial colony if given by the function $f(t) = 500e^{-.87t}$ where t is in hours and $f(t)$ is in thousands of cells.
Then $f(0) = 500e^{-.087(0)} = 500e^0 = 500$, so there are initially 500,000 cells in the colony.
Similarly, $f(5) = 500e^{-.087(5)} = 500e^{-0.435} \approx 323.632$, so after 5 hours, the population of the colony has been reduced to 323,632 cells.
The Derivative of the Exponential Function: The basic rule for differentiating the exponential function is:
$\frac{d}{dx}e^x = e^x$
We'll not prove this rule, but it is in fact true that the exponential function is its own derivative!

The Chain Rule for Exponentials:

$\frac{d}{dx}e^{f(x)} = f'(x)\cdot e^{f(x)}$

Examples of Derivatives Involving Exponential Functions:

1. If $f(x) = e^{3x}$, then, using the Chain Rule for Exponentials: $f'(x) = e^{3x}\cdot(3) = 3e^{3x}$.
2. If $g(x) = e^{x^2}$, then, using the Chain Rule for Exponentials: $g'(x) = e^{x^2}\cdot(2x) = 2xe^{x^2}$.
3. If $h(x) = x^2e^{5x}$, then, by the product rule: $h'(x) = 2xe^{5x} + 5x^2e^{5x}$.
4. If $k(x) = \left(e^{2x}+3x^2\right)^{\frac{5}{2}} = \frac{5}{2}\left(e^{2x}+3x^2\right)^{\frac{3}{2}}\left(2e^{2x}+6x\right) = \left(5e^{2x}+15x\right)\left(e^{2x}+3x^2\right)^{\frac{3}{2}}$
5. If $\ell(x) = e^{e^{x^2}}$, then, applying the Chain Rule for Exponentials several times: $\ell'(x) = e^{e^{x^2}}\cdot e^{x^2}\cdot x^2$

Applications of the Derivative Involving Exponentials:

1. Find the slope of the tangent line to $f(x) = 1 - e^{2x}$ at the point where f crosses the x-axis. Then find the equation of the tangent line.

   First notice that if $f(x) = 0$, then $0 = 1 - e^{2x}$, so $e^{2x} = 1$, or, $\ln(e^{2x}) = \ln(1)$.

   Therefore, $2x = 0$, so $x = 0$. Therefore, the point of tangency is $(0, 0)$.

   Next, $f'(x) = -2e^{2x}$, so $m = f'(0) = -2e^0 = -2(1) = -2$.

   Thus, the tangent line has equation $y = -2x$.