Math 229

Integration Handout

Antiderivatives and Integration

A. Antiderivatives

A function F is an **antiderivative** of f if $F'(x) = f(x)$.

Example: $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$, since $F'(x) = 3x^2 = f(x)$.

Note: If $F(x)$ is an antiderivative of f, then so is $F(x) + C$ for any constant C.

B. Indefinite Integrals

The notation $\int f(x)\ dx = F(x) + C$ is used to denote the indefinite integral of the function $f(x)$ with respect to the variable x. Here, $F(x) + C$ is the general antiderivative of $f(x)$.

C. Integration Rules

1. Integrating a constant: $\int a\ dx = ax + C$ for any constant a.

   Example: $\int \frac{3}{2}\ dx = \frac{3}{2}x + C$

2. Integrating Power Functions: $\int x^r\ dx = \frac{1}{r+1}x^{r+1} + C$

   Example: $\int x^{\frac{3}{2}}\ dx = \frac{2}{5}x^{\frac{5}{2}} + C$

3. Constant Multiples: $\int af(x)\ dx = a\int f(x)\ dx$ for any constant a.

   Example: $\int 7x^{\frac{3}{2}}\ dx = 7\int x^{\frac{3}{2}}\ dx = (7)\frac{2}{5}x^{\frac{5}{2}} + C = \frac{14}{5}x^{\frac{5}{2}} + C$

4. Sums and Differences: $\int f(x) + g(x)\ dx = \int f(x)\ dx + \int g(x)\ dx$,

   and $\int f(x) - g(x)\ dx = \int f(x)\ dx - \int g(x)\ dx$ for any functions f and g.

   Example: $\int 3x^2 + x^{\frac{3}{2}}\ dx = \int 3x^2\ dx + \int x^{\frac{3}{2}}\ dx = x^3 + \frac{2}{5}x^{\frac{5}{2}} + C$

5. Integrals Resulting in Exponentials and Natural Logarithms:

   $\int e^x\ dx = e^x + C$, and $\int \frac{1}{x}\ dx = \ln|x| + C$.

D. An Initial Value Problem

Suppose $f'(x) = 5x^2 - 4x + 7$ and $f(1) = 5$. Find $f(x)$.

Antidifferentiating, $f(x) = \frac{5}{3}x^3 - 2x^2 + 7x + C$.

Then $f(1) = 5 = \frac{5}{3}(1)^3 - 2(1)^2 + 7(1) + C = \frac{5}{3} - 2 + 7 + C$

Therefore, $5 - \frac{5}{3} - 5 = C$, so $C = -\frac{5}{3}$.

Hence $f(x) = \frac{5}{3}x^3 - 2x^2 + 7x - \frac{5}{3}$.

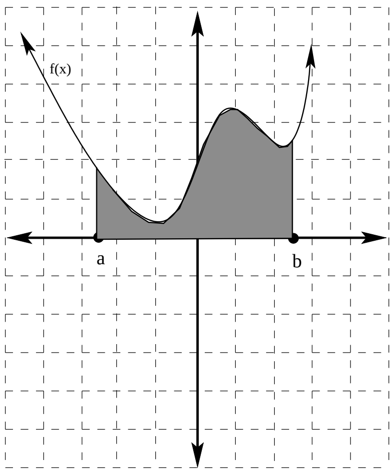
E. Definite Integrals

The Fundamental Theorem of Calculus:

Let f be a continuous function on the interval $[a, b]$, and let F be an antiderivative of f.

Then $\int_a^b f(x)\ dx = F(b) - F(a)$

Note: The number resulting from this computation is the area "under" the graph of $f(x)$ between $x = a$ and $x = b$.



Example: $\int_1^3 3x^2\ dx = x^3 \Big|_1^3 = 3^3 - 1^3 = 27 - 1 = 26$

F. Properties of Definite Integrals

1. $\int_a^a f(x)\ dx = 0$
2. $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$
3. $\int_a^b cf(x)\ dx = c\int_a^b f(x)\ dx$, for any constant c
4. $\int_a^b f(x) \pm g(x)\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$
5. $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$, for any $a < c < b$

G. The Average Value of a Function

Let f be a function that is integrable on an interval $[a, b]$. Then the **average value** of f over $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\ dx$.