Math 229
Logarithms

Definition: The **Logarithm of** x **to the base** b is defined as follows: $y = \log_b x$ if and only if $x = b^y$. for $x > 0$ and $b > 0, b \neq 1$. A logarithm basically asks: "what power would I need to raise the base b to in order to get x as the result?"

Examples:
(a) $\log_2 8 = 3$ since $8 = 2^3$
(b) $\log_2 \frac{1}{2} = -1$ since $\frac{1}{2} = 2^{-1}$
(c) $\log_3 81 = 4$ since $81 = 3^4$
(d) $\log_8 \frac{1}{64} = -2$ since $\frac{1}{64} = 8^{-2}$

More Examples:
(a) Suppose $\log_5 x = 3$. Find x.
Since $\log_5 x = 3$, $x = 5^3 = 125$.
(b) Suppose $\log_z 16 = 2$. Find z.
Since $\log_z 16 = 2$, $z^2 = 16$, so $z = \pm 4$. But since we know that $z > 0$, then $z = 4$.

Notation:
(1) If $b = 10$, we abbreviate $\log_{10} x$ as $\log x$.
(2) If $b = e$, we abbreviate $\log_e x$ as $\ln x$.

Properties of logarithms: Let m and n be positive real numbers.

1. $\log_b mn = \log_b m + \log_b n$
2. $\log_b \frac{m}{n} = \log_b m - \log_b n$
3. $\log_b m^n = n \cdot \log_b m$
4. $\log_b 1 = 0$
5. $\log_b b = 1$
6. $\log_b b^x = x$
7. $b^{\log_b x} = x$

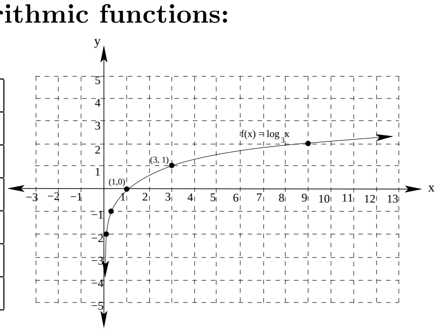
Examples:

1. $\log 16 = \log 4^2 = 2 \log 4$
2. $\log \frac{7}{16} = \log 7 - \log 16 = \log 7 - \log 4^2 = \log 7 - 2 \log 4$
3. $\log 24 = (\log 8 \cdot 3) = \log 8 + \log 3 = log 2^3 + \log 3 = 3 \log 2 + \log 3$
4. $\ln \left(\frac{e^{2x}}{e^x} \right) = \ln e^2 x - \ln e^x = 2x \ln e - x \ln e = 2x - x = x$

Graphs of logarithmic functions:

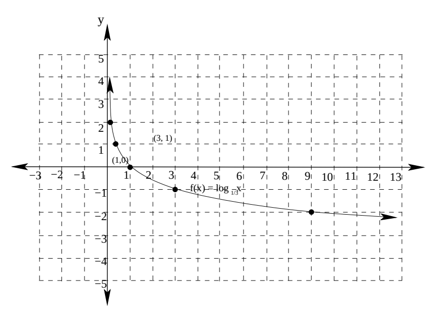
$f(x) = \log_3 x$

x	$f(x)$
0	undefined
1	0
3	1
9	2
$\frac{1}{3}$	-1
$\frac{1}{9}$	-2



$f(x) = \log_{\frac{1}{3}} x$

x	$f(x)$
0	undefined
1	0
$\frac{1}{3}$	1
$\frac{1}{9}$	2
3	-1
9	-2



Properties of Logarithmic Graphs:

1. Domain: $(0, \infty)$
2. Range: $(-\infty, \infty)$
3. y-intercept: none. x intercept $(1, 0)$
4. Continuous on $(0, \infty)$
5. Increasing if $b > 1$. Decreasing if $0 < b < 1$.

The Derivative of Logarithmic Functions: The basic rule for differentiating the natural logarithmic function is:

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Proof:

Recall that by the inverse property of exponentials and logarithms, $e^{\ln x} = x$ for $x > 0$.
Differentiating both sides of this equation, $\frac{d}{dx}\left(e^{\ln x}\right) = \frac{d}{dx}x$
Which, by the chain rule, is: $\frac{d}{dx}(\ln x) \cdot e^{\ln x} = 1$, or, again applying the inverse property: $\frac{d}{dx}(\ln x) \cdot x = 1$.
Therefore, dividing both sides by x: $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

The Chain Rule for Logarithmic Functions:

$$\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$$

Examples of Derivatives Involving Logarithmic Functions:

1. If $f(x) = x^2 \ln x$, then, by the product rule: $f'(x) = 2x\ln x + x^2 \cdot \frac{1}{x} = 2x\ln x + x$.
2. If $g(x) = \ln(x^2)$, then, using the Chain Rule: $g'(x) = \frac{2x}{x^2} = \frac{2}{x}$.

   Alternatively, we could have used the properties of logarithms to rewrite $g(x) = \ln(x^2)$ as $g(x) = 2\ln x$. Then $g'(x) = 2 \cdot \frac{1}{x} = \frac{2}{x}$.
3. If $h(x) = \ln[(x^2+1)(3x-2)^3]$, then, again using the properties of logarithms to rewrite $h(x)$,

   $h(x) = \ln(x^2+1) + \ln(3x-2)^3 = \ln(x^2+1) + 3\ln(3x-2)$.

   Thus $h'(x) = \frac{2x}{x^2+1} + 3 \cdot \frac{3}{3x-2} = \frac{2x}{x^2+1} + \frac{9}{3x-2}$.