The Algebra of Matrices

A. Addition and Subtraction:
Given two matrices that are precisely the same size, we add the matrices by adding and subtracting each corresponding entry. Similarly, we subtract the matrices by subtracting each corresponding entry.

Examples:

1. $\begin{bmatrix} 3 & 2 & 5 \\ 4 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 5 & -2 & 0 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 5 \\ 7 & 2 & 2 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 2 & 5 \\ 4 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 5 & -2 & 0 \\ 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 5 \\ 1 & -2 & -6 \end{bmatrix}$

B. Scalar Multiplication:
Given a matrix of *any* size and a real number, we multiply (or *scale*) the matrix by the given real number (or *scalar*) by multiplying each entry of the matrix by the given constant.

Examples:

1. Let $A = \begin{bmatrix} 3 & 2 & 5 \\ 4 & 0 & -2 \end{bmatrix}$. Then $3A = 3 \cdot \begin{bmatrix} 3 & 2 & 5 \\ 4 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 9 & 6 & 15 \\ 12 & 0 & -6 \end{bmatrix}$

2. Combining subtraction and scalar multiplication: Let $A = \begin{bmatrix} 3 & 2 & 5 \\ 4 & 0 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 & 0 \\ 3 & 2 & 4 \end{bmatrix}$.

Then $2A - 3B = 2 \cdot \begin{bmatrix} 3 & 2 & 5 \\ 4 & 0 & -2 \end{bmatrix} - 3 \cdot \begin{bmatrix} 5 & -2 & 0 \\ 3 & 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 6 & 4 & 10 \\ 8 & 0 & -4 \end{bmatrix} - \begin{bmatrix} 15 & -6 & 0 \\ 9 & 6 & 12 \end{bmatrix} = \begin{bmatrix} -9 & 10 & 10 \\ -1 & -6 & -16 \end{bmatrix}$

C. Multiplying Square Matrices:
To multiply square matrices of the same size, we form a new matrix by "multiplying" the *rows* of the first matrix by the *columns* of the second matrix. The first row of the first matrix and the first column of the second matrix combine to give us the entry in the top left corner of our new matrix (position (1,1)). The first row and the second column combine to give us the next entry in the top row of out new matrix (position (1,2), and so on)

Example:
Let $A = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix}$.

Then $AB = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} (4)(2) + (-1)(0) & (4)(-3) + (-1)(5) \\ (2)(2) + (3)(0) & (2)(-3) + (3)(5) \end{bmatrix} = \begin{bmatrix} 8 & -17 \\ 4 & 9 \end{bmatrix}$

D: Multiplying Non-Square Matrices:
We can multiply *non*-square matrices only if the rows of the first matrix are the same "size" as the columns of the second matrix. We can multiply an $m \times k$ matrix by a $k \times n$ to obtain an $m \times n$ matrix. We combine rows of the first matrix with columns in the same way as we did for square matrices.

Example:
Let $A = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ -1 & 2 \\ 3 & 7 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & -2 \\ 4 & -5 \end{bmatrix}$.

Then $AB = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ -1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} (-1)(2) + (3)(-1) + (5)(3) & (-1)(4) + (3)(2) + (5)(7) \\ (2)(2) + (0)(-1) + (3)(3) & (2)(4) + (0)(2) + (3)(7) \end{bmatrix} = \begin{bmatrix} 10 & 37 \\ 13 & 29 \end{bmatrix}$

$AC = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 \\ 4 & -5 \end{bmatrix}$ is undefined since the size of the rows of A and columns of C are not the same.