Math 262
Exam 1 - Practice Problems

1. Find the area between the given curves:

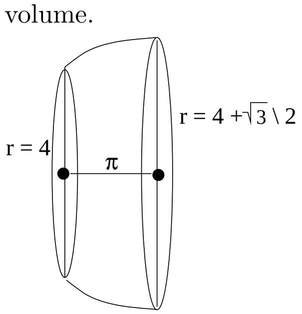
   (a) $y = x^2 + 1$ and $y = 3x - 1$
   First notice that these curves intersect when $x^2+1 = 3x-1$, or when $x^2-3x+2 = 0$. That is, when $(x-2)(x-1) = 0$, or when $x = 2$ and $x = 1$.
   Next, notice that $3x - 1 \geq x^2 + 1$ on $[1, 2]$. Thus the area between these curves is given by:
   $$\int_1^2 (3x-1) - (x^2+1)\ dx = \int_1^2 -x^2 + 3x - 2\ dx = -\frac{x^3}{3} + \frac{3}{2}x^2 - 2x\Big|_1^2 = \frac{1}{6}$$

   (b) $y = x^2 - 1$ and $y = 1 - x$ on $[0, 2]$
   First notice that these curves intersect when $x^2-1 = 1-x$, or when $x^2+x-2 = 0$. That is, when $(x+2)(x-1) = 0$, or when $x = -2$ and $x = 1$.
   Next, notice that $1 - x \geq x^2 - 1$ on $[0, 1]$ while $x^2 - 1 \geq 1 - x$ on $[1, 2]$. Thus the area between these curves is given by:
   $$\int_0^1 (1-x) - (x^2-1)\ dx + \int_1^2 (x^2-1) - (1-x)\ dx = \int_0^1 -x^2 - x + 2\ dx + \int_1^2 x^2 + x - 2\ dx$$
   $$= \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\Big|_0^1\right) + \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x\Big|_1^2\right) = 3.$$

   (c) $y = x$, $y = 2$, $y + x = 6$, and $y = 0$
   This integral is a bit easier if we slice the area horizontally and integrate with respect to y, so, solving for x in terms of y, we have $x = y$ and $x = 6 - y$, with $6 - y \geq y$ on $[0, 2]$. Therefore, the area of this region is given by:
   $$\int_0^2 (6-y) - (y)\ dy = \int_0^2 6 - 2y\ dy = 6y - y^2\Big|_1^2 = 12 - 4 = 8$$

   (d) $x = y^2$, $x = 4$
   We will find the area by integrating with respect to y. Notice that $y^2 = 4$ when $y = \pm 2$. Also, $4 \geq y^2$ on $[-2, 2]$. Therefore, the area of this region is given by:
   $$\int_{-2}^2 4 - y^2\ dy = 4y - \frac{1}{3}y^3\Big|_{-2}^2 = \frac{32}{3}.$$

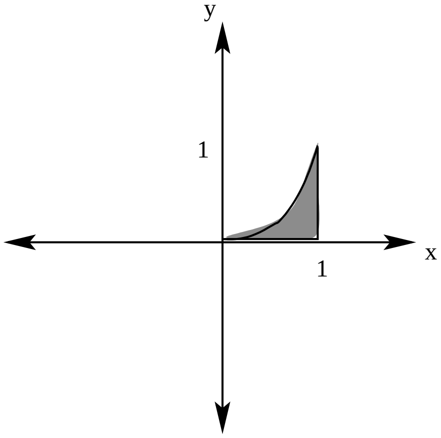
2. A pottery jar has circular cross sections or radius $4 + \sin(\frac{x}{3})$ inches for $0 \leq x \leq \pi$. Sketch the jar and then compute its volume.



$$V = \int_0^\pi \pi\left(4 + \sin(\frac{x}{3})\right)^2\ dx = \pi\int_0^\pi 16 + 8\sin\left(\frac{x}{3}\right) + \sin^2\left(\frac{x}{3}\right)\ dx = \pi\int_0^\pi 16 + 8\sin\left(\frac{x}{3}\right) + \left(\frac{1-\cos(\frac{2x}{3})}{2}\right)\ dx$$
$$= \pi\left[16x - 24\cos(\frac{x}{3}) + \frac{1}{2}x - \frac{3}{4}\sin(\frac{2x}{3})\Big|_0^\pi\right] = [16\pi - 24(\frac{1}{2}) + \frac{\pi}{2} - \frac{3}{4}\frac{\sqrt{3}}{2}] - [0 - 24 + 0 - 0] = \frac{33\pi}{2} + 12 - \frac{3\sqrt{3}}{8}.$$

3. Let R be the region bounded by $y = x^3$, $y = 0$, and $x = 1$

   (a) Sketch R and then find its area.



$$A = \int_0^1 x^3 \; dx = \frac{1}{4}x^4\Big|_0^1 = \frac{1}{4}.$$

   (b) Find the volume of the solid formed by rotating R about the x-axis.

   Integrating with respect to x and using disks, we see $V = \int_0^1 \pi \left(x^3\right)^2 \; dx = \pi \int_0^1 x^6 \; dx = \pi \left[\frac{1}{7}x^7\Big|_0^1\right] = \frac{\pi}{7}.$

   (c) Find the volume of the solid formed by rotating R about the y-axis.
   Integrating with respect to x and using cylindrical shells, we see:

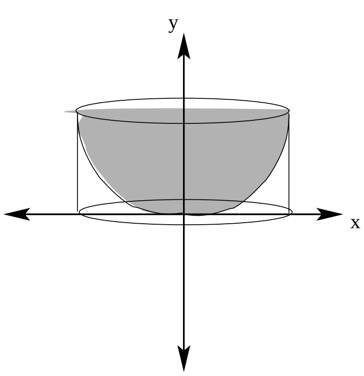
$$V = \int_0^1 2\pi x \left(x^3\right) \; dx = 2\pi \int_0^1 x^4 \; dx = 2\pi \left[\frac{1}{5}x^5\Big|_0^1\right] = \frac{2\pi}{5}.$$

4. Let R be the region bounded by $y = ax^2$, $y = h$, and the y-axis (where a and h are positive constants). Compute the volume of the solid formed by revolving this region about the y-axis. Show that your answer is equal to half the volume of a cylinder of height h and radius $r = \sqrt{\frac{h}{a}}$. Sketch a picture illustrating the relationship between these two volumes.

   If we solve for x, $y = ax^2$ becomes $x = \sqrt{\frac{y}{a}}$ and we can represent the volume of this solid of revolution by slicing horizontally into disks:

$$V = \int_0^h \pi \left(\sqrt{\frac{y}{a}}\right)^2 \; dy = \frac{\pi}{a}\int_0^h y \; dy = \frac{\pi}{a}\left[\frac{1}{2}y^2\Big|_0^h\right] = \frac{\pi h^2}{2a}.$$

   Next, recall that the formula for the volume of a cylinder is: $V = \pi r^2 h = \pi(\sqrt{\frac{h}{a}})^2 h = \pi\left(\frac{h}{a}\right)h = \frac{\pi h^2}{a}$, so we can see that our answer is equal to half the volume of this cylinder.



5. The great pyramid at Giza is 500 feet high rising from a square base with side length 750 feet. Use integration to find the volume of this pyramid.

   We will find the volume of this pyramid by noticing that slicing it horizontally yields square cross sections. Moreover, if we think of the height of our slices with $0 \leq y \leq 500$, then the side length of the square cross sections is given by $s(y) = 750 - \frac{3}{2}y$. Therefore, the volume of the great pyramid of Giza is:

   $$V = \int_0^{500} \left(750 - \frac{3}{2}y\right)^2 \, dy = \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(750 - \frac{3}{2}y\right)^3 \Bigg|_0^{250} = -\frac{2}{9}\left[(0) - (750)^3\right] = 93,750,000 ft^3.$$

   Notice that this agrees with the result of the geometric formula $V = \frac{1}{3}b^2h = \frac{1}{3}(750)^2(500) = 93,750,000ft^3$.

6. Find the volume of the solid formed by revolving the region bounded by $y = x^2 - 4$ and $y = 4 - x^2$ about the line $x = 2$.

   Notice that these two curves meet when $x = \pm 2$. Therefore, the volume of this solid of revolution is given by:

   $$V = \int_{-2}^{2} 2\pi(2-x)[(4-x^2)-(x^2-4)] \, dx = 2\pi \int_{-2}^{2} (2-x)(8-2x^2) \, dx = 2\pi \int_{-2}^{2} 2x^3 - 4x^2 - 8x + 16 \, dx$$

   $$= 2\pi \left[\frac{1}{2}x^4 - \frac{4}{3}x^3 - 4x^2 + 16x \Bigg|_{-2}^{2} \right] = 2\pi \left[(8 - \frac{32}{3} - 16 + 32) - (8 + \frac{32}{3} - 16 - 32) \right] = 2\pi[64 - \frac{64}{3}] = \frac{256\pi}{3}$$

7. Find the volume of the solid formed over the region bounded by $y = x^2 - 4$ and $y = 4 - x^2$ if its cross sections perpendicular to the y-axis are semicircles.

   First, since the semicircles run horizontally, we should integrate with respect to y. By symmetry with respect to the x-axis, we can integrate over $0 \leq y \leq 4$ and then double our result. Notice that the radius of each semicircular cross section is given by $r = \sqrt{4-y}$.

   Therefore, the volume of this region is given by:

   $$V = 2\int_0^4 \frac{\pi}{2}(4-y) \, dy = 2 \cdot \frac{\pi}{2}[4y - \frac{1}{2}y^2 \Bigg|_0^4] = \pi[16 - 8] = 8\pi.$$

8. Find the volume of the solid formed by revolving the region bounded by $y = x^2$ and the x-axis for $-1 \leq x \leq 1$ about the line $x = -2$.

   Based on the shape of this region, it is best to use cylindrical shells to find this volume. We do so using the following integral:

   $$V = \int_{-1}^{1} 2\pi(x+2)(x^2) \, dx = 2\pi \int_{-1}^{1} x^3 + 2x^2 \, dx = 2\pi[\frac{1}{4}x^4 + \frac{2}{3}x^3 \Bigg|_{-1}^{1}] = 2\pi[(\frac{1}{4} + \frac{2}{3}) - (\frac{1}{4} - \frac{2}{3})] = 2\pi(\frac{4}{3}) = \frac{8\pi}{3}.$$

9. Find the volume of the solid formed by revolving the region bounded by $x = y^2$ and $x = 1$ about the line $y = -2$.

   Based on the shape of this region, it is best to use cylindrical shells to find this volume. We do so using the following integral:

   $$V = \int_{-1}^{1} 2\pi(y+2)(1-y^2) \, dy = 2\pi \int_{-1}^{1} -y^3 - 2y^2 + y + 2 \, dy = 2\pi[-\frac{1}{4}y^4 - \frac{2}{3}y^3 + \frac{1}{2}y^2 + 2y \Bigg|_{-1}^{1}]$$

   $$= 2\pi[(-\frac{1}{4} - \frac{2}{3} + \frac{1}{2} + 2) - (-\frac{1}{4} + \frac{2}{3} + \frac{1}{2} - 2)] = 2\pi(4 - \frac{4}{3}) = \frac{16\pi}{3}.$$

10. Find the volume of the solid formed by revolving the region bounded by $x = y^2$ and $x = 2 + y$ about:

First, notice that these two curves intersect when $y^2 = 2 + y$, or when $y^2 - y - 2 = 0$. That is when $(y - 2)(y + 1) = 0$. Therefore, the two points of intersection are $(1, -1)$ and $(4, 2)$.

(a) the line $x = -1$.

Notice that on $[0, 1]$ the region enclosed by these curves is bounded above and below by the parabola $x = y^2$ while on $[1, 4]$ the region is bounded above by $x = y^2$ and below by $x = 2 + y$. Because of this, in order to find the volume of this solid, we must consider these two subintervals separately. Therefore, the volume of this solid is given by:

$$V = \int_0^1 2\pi(x + 1)(2\sqrt{x})\ dx + \int_1^4 2\pi(x + 1)(\sqrt{x} - (x - 2))\ dx$$

$$= 4\pi \int_0^1 x^{\frac{3}{2}} + x^{\frac{1}{2}}\ dx + 2\pi \int_1^4 -x^2 + x^{\frac{3}{2}} + x^{\frac{1}{2}} + x + 2\ dx =$$

$$4\pi[\frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}}\Bigg|_0^1] + 2\pi[-\frac{1}{3}x^3 + \frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 + 2x\Bigg|_1^4]$$

$$= 4\pi[\frac{2}{5} + \frac{2}{3}] + 2\pi[(\frac{2}{5}32 + \frac{2}{3}8 - \frac{1}{3}64 + \frac{1}{2}16 + 8) - (\frac{2}{5} + \frac{2}{3} - \frac{1}{3} + \frac{1}{2} + 2)] = \frac{117\pi}{5}$$

[Note: It is actually easier to do this part using washers – see part (c) below. I just wanted to show both possibilities]

(b) the line $y = -1$.

Here we view $x = 2 + y$ as the right function, $x = y^2$ as the left function bounding our region and we integrate using cylindrical shells:

$$V = \int_{-1}^2 2\pi(y + 1)[(2 + y) - (y^2)]\ dy = 2\pi \int_{-1}^2 -y^3 + 3y + 2\ dy = 2\pi[-\frac{1}{4}y^4 + \frac{3}{2}y^2 + 2y\Bigg|_{-1}^2]$$

$$= 2\pi[(-4 + 6 + 4) - (-\frac{1}{4} + \frac{3}{2} - 2)] = \frac{27\pi}{2}$$

(c) the line $x = -2$.

We could do this similarly to part (a) above, but it is actually simpler to do this using washers:

$$V = \pi \int_{-1}^2 ((2 + y) + 2)^2 - (y^2 + 2)^2\ dy = \pi \int_{-1}^2 (y^2 + 8y + 16) - (y^4 + 4y^2 + 4)\ dy = \pi \int_{-1}^2 -y^4 - 3y^2 + 8y + 12\ dy$$

$$= \pi[-\frac{1}{5}y^5 - y^3 + 4y^2 + 12y\Bigg|_{-1}^2] = \pi[(-\frac{32}{5} - 8 + 16 + 24) - (\frac{1}{5} + 1 + 4 - 12)] = \frac{162\pi}{5}$$

11. Set up the integral for the arc length of each of the following curves on the given interval. You DO NOT need to evaluate the integral.

(a) $y = x^3 + x$ on $[-2, 5]$

Notice that $y' = 3x^2 + 1$ and $(y')^2 = 9x^4 + 6x^2 + 1$

Therefore, $L = \int_{-2}^5 \sqrt{9x^4 + 6x^2 + 2}\ dx$

(b) $y = \tan x$ on $[0, \frac{\pi}{4}]$

$y' = \sec^2 x$ and $(y')^2 = \sec^4 x$

Therefore, $L = \int_0^{\frac{\pi}{4}} \sqrt{\sec^4 x + 1}\ dx$

(c) $y = \frac{1}{x^2+1}$ on $[0, 2]$

Notice that $y' = \dfrac{-2x}{(x^2 + 1)^2}$ and $(y')^2 = \dfrac{4x^2}{(x^2 + 1)^4}$

Therefore, $L = \int_0^2 \sqrt{1 + \dfrac{4x^2}{(x^2 + 1)^4}}\ dx$

(d) $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Notice that this curve is an ellipse centered at the origin with major axis of length 8 and minor axis of length 6.

Solving for y, we obtain: $\frac{y^2}{9} = 1 - \frac{x^2}{16}$, so $y^2 = 9 - \frac{9x^2}{16}$ or $y = \pm\sqrt{\dfrac{144 - 9x^2}{16}}$

If we take the positive half of this equation, we can find the arc length of the ellipse and double it in order to obtain the arc length of the entire ellipse.

Then $y' = \frac{1}{2}\left(\frac{144-9x^2}{16}\right)^{-\frac{1}{2}}\left(-\frac{18x}{16}\right) = -\frac{9x}{16}\left(\frac{144-9x^2}{16}\right)^{-\frac{1}{2}}$ and $(y')^2 = \frac{81x}{256}\left(\frac{144-9x^2}{16}\right)^{-1} = \frac{20736x^2}{144-9x^2}$

Therefore, $L = 2\int_{-4}^{4}\sqrt{1+\frac{20736x^2}{144-9x^2}}\;dx$

12. Set up the integral for the surface area of each of the following on the given interval. You DO NOT need to evaluate the integral.

(a) $y = \sin x$ on $[0,\pi]$ revolved about the x-axis.

$S = \int_0^{\pi} 2\pi r\;ds = 2\pi\int_0^{\pi}(\sin x)\sqrt{1+\cos^2 x}\;dx$

(b) $y = x^3 - 1$ on $[1,2]$ revolved about the x-axis.

$S = \int_1^2 2\pi r\;ds = 2\pi\int_1^2 (x^3-1)\sqrt{1+9x^4}\;dx$

(c) $y = x^3 - 1$ on $[1,2]$ revolved about the y-axis.

Since we are revolving the region about the y-axis, we need to solve the equation or our curve for x: $y+1 = x^3$, so $x = (y+1)^{\frac{1}{3}}$. Therefore, $x' = \frac{1}{3}(y+1)^{-\frac{2}{3}}$

Then $S = \int_0^7 2\pi r\;ds = 2\pi\int_0^7 (y+1)^{\frac{1}{3}}\sqrt{1+\frac{1}{9}(y+1)^{-\frac{4}{3}}}\;dy$

13. A force of 10 pounds stretches a spring 2 inches. Find the work done in stretching this spring 3 inches beyond its natural length (give your answer in ft-lbs).

Recall that By Hooke's Law, the force needed to stretch of compress a spring is given by $F(x) = kx$. Here, $F(\frac{2}{12}) = k\left(\frac{1}{6}\right) = 10$, so $k = 60$.

Therefore, $W = \int_0^{\frac{3}{12}} 60x\;dx = 30x^2\Big|_0^{\frac{1}{4}} = 30\left(\frac{1}{4}\right)^2 = \frac{15}{8}ft.-lbs.$

14. A cylindrical tank is resting on the ground with its axis vertical; it has a radius of 5 feet and a height of 10 feet. Find the amount of work done in filling this tank with water pumped in from ground level (use $\rho = 62.5/ft^3$ for the density of water).

$W = F\cdot d = \int_0^{10}\pi 5^2 h(62.5)\;dh = 25\pi(62.5)\frac{1}{2}h^2\Big|_0^{10} = 25\pi(62.5)(50) = 78125\pi ft.-lbs.$

15. A water tank is in the shape of a right circular cone of altitude 10 feet and base radius 5 feet, with its vertex one the ground (think of an ice cream cone with its point facing down). If the tank is half full, find the work done in pumping all of the water out the *top* of the tank.

If we slice the center of the cone vertically, we obtain a right triangular cross section with height 10 feet and width 5 feet. Using similar triangles, if we move up the cone to a height h, then the horizontal cross section at height h has radius r satisfying $\frac{h}{10} = \frac{r}{5}$, so $r = \frac{h}{2}$. Therefore, the total work in pumping all of the water out of the top of the cone is given by:

$$W = \int_0^5 \pi(\frac{h}{2})^2(62.5)(10-h)\;dh = (62.5\pi)\int_0^5 \frac{5}{2}h^2 - \frac{1}{4}h^3\;dh = (62.5\pi)[\frac{5}{6}h^3 - \frac{1}{16}h^4\Big|_0^5] = (62.5\pi)[\frac{625}{6} - \frac{625}{16}] = \frac{390625\pi}{96}$$

16. A chain of density 5 kilograms per meter is hanging down the side of a building that is 100 meters tall. How much work is done if this chain is used to lift a 500 kilogram mass 100 feet up the side of the building?

First recall that the downward force of gravity in a $1kg$ mass is $9.8\frac{m}{sec^2}$. Next, $100ft \approx (100)(.3048\frac{m}{ft}) = 30.48 meters$.

Therefore, the work done in using this chain to lift a $500kg$ mass is given by the integral:

$$W = \int_0^{30.48}(9.8)(500+(5)(100-y))\;dy = (9.8)\int_0^{30.48} 1000 - 5y\;dy = (9.8)[1000y - \frac{5}{2}y^2\Big|_0^{30.48}]$$

$= (9.8)[30480 - 2322.576] = 275,942.7552 Joules.$

17. Find the center of mass of the following systems of point masses:

   (a) A 50 kg mass 5 units left of the x-axis, a 30 kg mass 1 unit left of the x-axis and a 20 kg mass 12 units right of the x-axis.
   $m = 50 + 30 + 20 = 100$ while $M_0 = (50)(-5) + (30)(-1) + (20)(12) = -40$
   Therefore, $\overline{x} = \frac{-40}{100} = -.4$

   (b) A 40 kg mass at the point (-2, 5), a 25 kg mass at the point (3, -1) and a 10 lb mass at the point (5, 7).
   Notice that $10lb = (10)(.4536kg/lb) = 4.536kg$
   $m = 40 + 25 + 4.536 = 69.536kg$
   $M_y = (40)(-2) + (25)(3) + (4.536)(5) = 17.68$ while $M_x = (40)(5) + (25)(-1) + (4.536)(7) = 206.752$
   Therefore, $\overline{x} = \frac{M_y}{m} = \frac{17.68}{69.536} = 0.2543$ while $\overline{y} = \frac{206.752}{69.536} = 2.9733$

18. Compute both the mass and the center of mass of a steel rod with density $\rho(x) = 3 - \frac{x}{6}$ for $0 \leq x \leq 6$.

   The mass of this bar is given by: $m = \displaystyle\int_0^6 \rho(x)\ dx = \int_0^6 3 - \frac{x}{6}\ dx = 3x - \frac{1}{12}x^2\Big|_0^6 = 18 - \frac{36}{12} = 15kg.$

   $$M_0 = \int_0^6 x\rho(x)\ dx = \int_0^6 3x - \frac{x^2}{6}\ dx = \frac{3}{2}x^2 - \frac{1}{18}x^3\Big|_0^6 = 54 - 12 = 42.$$

   Therefore, $\overline{x} = \frac{M_0}{m} = \frac{42}{15} = 2.8$ meters from the left end of the rod.

19. Find the centroid of the region R bounded by $y = 4 - x^2$ and the x-axis.
   First notice that, by symmetry, $\overline{x} = 0$. Also, throughout this problem, we will take $\rho = 1$.

   Then $m = \displaystyle\int_{-2}^2 \rho(4 - x^2)\ dx = 2\int_0^2 4 - x^2\ dx = 2[4x - \frac{1}{3}x^3\Big|_0^2] = 2[8 - \frac{8}{3}] = \frac{32}{3}$

   Next, $M_x = \displaystyle\int_{-2}^2 \frac{1}{2}(4 - x^2 + (0))(\rho)((4 - x^2) - (0))\ dx = \int_0^2 16 - 8x^2 + x^4\ dx = 16x - \frac{8}{3}x^2 + \frac{1}{5}x^5\Big|_0^2$

   $= 32 - \dfrac{64}{3} + \dfrac{32}{5} = \dfrac{256}{15}.$

   Therefore, $\overline{y} = \dfrac{\frac{256}{15}}{\frac{32}{3}} = \dfrac{8}{5}$

   Hence the center of mass is at $(0, \frac{8}{5})$

20. Find the centroid of the isosceles triangle formed by the points $(0, 0)$, $(6, 0)$ and $(3, 9)$
   This problem is much easier to do is we think of shifting the coordinate system for this triangle left by 3 units, centering the triangle on the y-axis and making its points $(-3, 0)$, $(3, 0)$ and $(0, 9)$
   First, notice that if we assume $\rho = 1$, then $m = A \cdot \rho = \frac{1}{2}bh = \frac{1}{2}(6)(9) = 27$
   For our shifted triangle, we see by symmetry that $\overline{x} = 0$.
   Next, we find the equation for the line containing $(3, 0)$ and $(0, 9)$:
   It has slope: $\frac{9}{-3} = -3$, so the equation is $y = -3x + 9$

   Then $M_x = \displaystyle\int_{-3}^3 \frac{1}{2}(f(x) + g(x))(\rho)(f(x) - g(x))\ dx = 2\int_0^3 \frac{1}{2}(-3x + 9 + (0))(-3x + 9 - (0))\ dx$

   $= \displaystyle\int_0^3 9x^2 - 54x + 81\ dx = 3x^3 - 27x^2 + 81x\Big|_0^3 = 81 - 243 + 243 = 81.$

   Therefore, $\overline{y} = \frac{81}{27} = 3$
   Hence the center of mass of the *shifted* triangle is at $(0, 3)$.
   Shifting this right 3 units, the center of mass of the *original* triangle is at $(3, 3)$.