Math 262
Exam 2 - Practice Problems

1. Determine whether or not each of the following functions is one-to-one.

   (a) $f(x) = \ln(x^2)$
   Notice that $f(2) = f(-2) = \ln 4$. Therefore, this function is not one-to-one.

   (b) $f(x) = x^5 + 2x^3 - 2$
   First notice that f is continuous. Next, $f'(x) = 5x^4 + 6x^2$, which is always positive. Therefore, f is a continuous function which is increasing throughout is domain. Thus $f(x)$ is a one-to-one function.

2. Determine whether or not each of the following functions is one-to-one. If it is, find the inverse function.

   (a) $f(x) = x^3 + 4$
   First notice that f is continuous. Next, $f'(x) = 3x^2$, which is always non-negative. Therefore, f is a continuous function which is increasing throughout is domain. Thus $f(x)$ is a one-to-one function.
   Then, solving $y = x^3 + 4$, $y - 4 = x^3$, or $\sqrt[3]{y-4} = x$. Thus $f^{-1}(x) = \sqrt[3]{x-4}$.
   Check: $f(\sqrt[3]{x-4}) = (\sqrt[3]{x-4})^3 + 4 = x - 4 + 4 = x$ and $f^{-1}(x^3+4) = \sqrt[3]{(x^3+4) - 4} = \sqrt[3]{x^3} = x$.

   (b) $f(x) = x^4 - 2$
   Notice that $f(1) = f(-1) = -1$. Therefore, f is not a one-to-one function on its full domain.

   (c) $f(x) = \dfrac{x+3}{2x-1}$
   Suppose that $f(a) = f(b)$ for some pair of real numbers $a, b (\neq \frac{1}{2})$. Then $\dfrac{a+3}{2a-1} = \dfrac{b+3}{2b-1}$.
   Then $(a+3)(2b-1) = (b+3)(2a-1)$, so $2ab + 6b - a - 3 = 2ab + 6a - b - 3$.
   Therefore, $7b = 7a$, or $b = a$. Hence $f(x)$ is one-to-one.
   To find the inverse of $f(x)$, we solve $y = \dfrac{x+3}{2x-1}$ for x.
   Then $y(2x-1) = x + 3$, or $2xy - y = x + 3$. Thus $2xy - x = y + 3$, or $x(y-1) = y + 3$
   Hence $x = \dfrac{y+3}{2y-1}$, so $f^{-1}(x) = \dfrac{x+3}{2x-1}$
   Check: $f(\dfrac{x+3}{2x-1}) = \dfrac{\frac{x+3}{2x-1} + 3}{2\left(\frac{x+3}{2x-1}\right) - 1} = \dfrac{\frac{(x+3)+3(2x-1)}{2x-1}}{\frac{2(x+3)-(2x-1)}{2x-1}} = \dfrac{7x}{2x-1} \cdot \dfrac{2x-1}{7} = x$

3. (4 points each) Given the following table of values and that $g(x) = f^{-1}(x)$:

	$f(x)$	$f'(x)$
0	4	-1
1	2	3
2	1	5
3	-5	7
4	3	-2

Find the following:

(a) $g(2) = f^{-1}(2) = 1$

(b) $g(4) = f^{-1}(4) = 0$

(c) $g'(4) = \dfrac{1}{f'(g(4))} = \dfrac{1}{f'(0)} = -1$

(d) $g'(3) = \dfrac{1}{f'(g(3))} = \dfrac{1}{f'(4)} = -\dfrac{1}{2}$

4. *Without* solving for the inverse function $f^{-1}(x)$, find the derivative of the inverse function at the given x value:

   (a) $f(x) = x^3 + 2x + 1$, $x = 1$.
   $f'(x) = 3x^2 + 2$. Also, since $f(0) = 1$, $f^{-1}(1) = 0$, and $f'(0) = 2$.
   Therefore, by Corollary 7.8, $D_x\left(f^{-1}(1)\right) = \frac{1}{f'(0)} = \frac{1}{2}$.

   (b) $f(x) = \sqrt{x^3 + 2x + 1}$, $x = 2$
   $f'(x) = \frac{1}{2}(x^3 + 2x + 1)^{-\frac{1}{2}}(3x^2 + 2) = \frac{3x^2+2}{2\sqrt{x^3+2x+1}}$.
   Also, since $f(1) = \sqrt{4} = 2$, $f^{-1}(2) = 1$, and $f'(1) = \frac{5}{2\sqrt{4}} = \frac{5}{4}$.
   Therefore, by Corollary 7.8, $D_x\left(f^{-1}(2)\right) = \frac{1}{f'(1)} = \frac{4}{5}$.

5. Find the derivative of each of the following functions:

   (a) $f(x) = \ln(\sec x + \tan x)$
   $f'(x) = \frac{1}{\sec x+\tan x} \cdot (\sec x \tan x + \sec^2 x) = \frac{\sec x(\tan x+\sec x)}{\sec x+\tan x} = \sec x$

   (b) $f(x) = \ln\sqrt{\dfrac{x^3}{x^5+1}}$
   First, $f(x) = \frac{1}{2}\left[\ln(x^3) - \ln(x^5+1)\right] = \frac{3}{2}\ln x - \frac{1}{2}\ln(x^5+1)$
   Therefore, $f'(x) = \frac{3}{2x} - \frac{5x^4}{2(x^5+1)}$

   (c) $f(x) = (x^5 - 4x^3 - 7)^8(1 - x^2)^7(5x + 10)^{12}$
   Using logarithmic differentiation, let $g(x) = \ln\left[(x^5 - 4x^3 - 7)^8(1 - x^2)^7(5x + 10)^{12}\right]$
   $= 8\ln(x^5 - 4x^3 - 7) + 7\ln(1 - x^2) + 12\ln(5x + 10)$.
   Then $g'(x) = \frac{8(5x^4-12x^2)}{x^5-4x^3-7} - \frac{7(2x)}{1-x^2} + \frac{12(5)}{5x+10}$.
   Hence $f'(x) = f(x)g'(x) = (x^5 - 4x^3 - 7)^8(1 - x^2)^7(5x + 10)^{12}\left[\frac{40x^4-96x^2)}{x^5-4x^3-7} - \frac{14x)}{1-x^2} + \frac{60}{5x+10}\right]$

   (d) $f(x) = e^{3x^2}\sec^{-1}(2x^3)$
   $f'(x) = 6xe^{3x^2}\sec^{-1}(2x^3) + e^{3x^2} \cdot \frac{6x^2}{2x^3\sqrt{4x^6-1}} = 6xe^{3x^2}\sec^{-1}(2x^3) + e^{3x^2} \cdot \frac{3}{x\sqrt{4x^6-1}}$

   (e) $f(x) = e^{\ln(\cos(\sqrt{x}))}$
   Notice that $f(x) = \cos(\sqrt{x})$, so $f'(x) = -\sin(\sqrt{x}) \cdot \frac{1}{2}x^{-\frac{1}{2}} = -\frac{\sin\sqrt{x}}{2\sqrt{x}}$.

   (f) $f(x) = x3^{-x}\log_5(7x^2)$
   $f'(x) = 3^{-x}\log_5(7x^2) - x\ln(3)3^{-x}\log_5(7x^2) + x3^{-x}\frac{14x}{(\ln 5)(7x^2)}$

6. Find all extrema and inflection points of the function $f(x) = x^2e^{-x}$
   $f'(x) = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2 - x)$, so $f(x)$ has critical numbers $x = 0$ and $x = 2$.
   $f''(x) = 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x} = 2e^{-x} - 4xe^{-x} + x^2e^{-x}$.
   If $f''(x) = 2e^{-x} - 4xe^{-x} + x^2e^{-x} = 0$, then $2 - 4x + x^2 = 0$.
   Applying the quadratic formula, this has solutions: $x = \frac{4\pm\sqrt{16-4(1)(2)}}{2} = 2 \pm sqrt2$
   Therefore, f has extrema at $(0, 0)$ and $(2, \frac{4}{e^2})$ and inflection points at $(2 + \sqrt{2}, \frac{6+4\sqrt{2}}{e^{2+\sqrt{2}}})$ and $(2 - \sqrt{2}, \frac{6-4\sqrt{2}}{e^{2-\sqrt{2}}})$

7. Find an equation to the tangent line to the graph of $ye^{2x} + x\ln|y| = 1 + \tan^{-1}(x)$ at the point $(0, 1)$.
   Differentiating implicitly, $y'e^{2x} + y(2e^{2x}) + \ln|y| + x \cdot \frac{1}{y} \cdot y' = \frac{1}{1+x^2}$
   Substituting $x = 0$ and $y = 1$, $y'(e^0) + 1(2e^0) + \ln(1) + (0) = \frac{1}{1}$, or $y' + 2 = 1$. Thus $y' = -1$.
   Then, by the point/slope formula: $y - 1 = -1(x - 0)$, or $y = -x + 1$ is the equation of the tangent line to the point $(0, 1)$.

8. Suppose that the population of a bacterial colony initially has 400 cells. After an hour, the population has increased to 800 cells. Find an equation for the population at any time. Then determine the population of the colony after 10 hours.

   First recall that $f(x) = Ae^{kt}$ is the general formula for exponential population growth. Next, $f(0) = Ae^0 = A = 400$.

   Also, $f(1) = 400e^k = 800$, so $2 = e^k$, and so $k = \ln 2$.

   Therefore, there are $f(10) = 400e^{10\ln(2)} = 409,600$ bacterial cells in the colony after 10 hours.

   Or you could just notice that since the population doubles every hour, the population after 10 hour is 2^{10} times the original population.

9. Scientists trying to date the age of a fossil estimate that it contains 20% of the carbon-14 originally present. Given that the half-life of carbon-14 is 5730 years, approximately how old is the fossil?

   First recall that $f(x) = Ae^{kt}$ is the general formula for exponential decay.

   Also, since the half-life of Carbon-14 is 5730 years, $e^{5730k} = \frac{1}{2}$, so $.5 = e^{5730k}$, and so $k = \frac{\ln .5}{5730}$.

   Therefore, to find the age of the fossil, we note that $.2A = Ae^{kt}$ or $.2 = e^{kt}$.

   Thus $\ln .2 = kt$, or $\ln .2 = \frac{\ln .5}{5730}t$. Hence $t = \frac{5730 \ln .2}{\ln .5} \approx 13,305$ years.

10. According to Newton's Law of Cooling, the temperature of an object placed in a room with ambient temperature T_a satisfies the differential equation $y'(t) = k[y(t) - T_a]$.

    (a) Use separation of variables to find a general solution to this differential equation.
    Let $y(t) = T$ and $y'(t) = \frac{dT}{dt}$. Then this differential equation can be written: $\frac{dT}{dt} = k(T - T_a)$.
    Then, separating the variables, $\displaystyle\int \frac{1}{T - T_a}\, dT = \int k\, dt.$
    That is, $\ln|T - T_a| = kt + C$, or $T - T_a = \pm e^{kt+C} = \pm e^{kt} \cdot e^C$.
    If we let $A = \pm e^C$, then $y(t) = T = Ae^{kt} + T_a$ is the general solution to this differential equation.

    (b) Suppose a bowl of porridge initially at 200°F (too hot) is placed in a 70°F room. One minute later, the porridge has cooled down to 180°F. How long will it take for the porridge to cool down to 120°F (just right)?
    Here, $T(0) = 200$°F, and $T_a = 70$°F. Then $y(0) = 200 = Ae^0 + 70$, so $A = 130$.
    Next, $y(1) = 130e^k + 70 = 180$, so $130e^k = 110$, so $e^k = \frac{110}{130}$, so $k = \ln \frac{11}{13}$.
    Therefore, $120 = 130e^{\ln \frac{11}{13} \cdot t} + 70$, so $\frac{50}{130} = e^{\ln \frac{11}{13} \cdot t}$.
    Hence $\ln \frac{5}{13} = \ln \frac{11}{13} \cdot t$, so $t = \frac{\ln \frac{5}{13}}{\ln \frac{11}{13}} \approx 5.72$ minutes.

11. Find each limit, (if it exists).

    (a) $\displaystyle\lim_{x \to 1} \frac{\sin(\pi x)}{x - 1}$
    Form: $\frac{0}{0}$, so, applying L'Hôpital's Rule:
    $$\lim_{x \to 1} \frac{\sin(\pi x)}{x - 1} = \lim_{x \to 1} \frac{\pi \cos(\pi x)}{1} = \frac{\pi(-1)}{1} = -\pi.$$

    (b) $\displaystyle\lim_{x \to 1} \frac{e^{x-1} - 1}{x^2 - 1}$
    Form: $\frac{0}{0}$, so, applying L'Hôpital's Rule:
    $$\lim_{x \to 1} \frac{e^{x-1} - 1}{x^2 - 1} = \lim_{x \to 1} \frac{e^{x-1}}{2x} = \frac{e^0}{2} = \frac{1}{2}.$$

    (c) $\displaystyle\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$
    Form: $\frac{\infty}{\infty}$, so, applying L'Hôpital's Rule:
    $$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-\frac{1}{2}}} = \lim_{x \to \infty} \frac{2x^{\frac{1}{2}}}{x} = \lim_{x \to \infty} \frac{2}{x^{\frac{1}{2}}} = 0$$

    (d) $\displaystyle\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right)$
    Form: $\infty \cdot$, so, before applying L'Hôpital's Rule, we rewrite this as:
    $\displaystyle\lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$, which has the form $\frac{0}{0}$, so, applying L'Hôpital's Rule:
    $$\lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x \to \infty} \frac{-x^{-2}\cos\left(\frac{1}{x}\right)}{-x^{-2}} = \lim_{x \to \infty} \frac{\cos\left(\frac{1}{x}\right)}{1} = 1$$

(e) $\lim_{x\to 0} \dfrac{x\sin x}{\cos x - 1}$

Form: $\frac{0}{0}$, so, applying L'Hôpital's Rule:

$\lim_{x\to 0} \dfrac{x\sin x}{\cos x - 1} = \lim_{x\to 0} \dfrac{\sin x + x\cos x}{-\sin x}$

This still has the form: $\frac{0}{0}$, so, applying L'Hôpital's Rule again:

$= \lim_{x\to 0} \dfrac{\cos x + \cos x - x\sin x}{-\cos x} = \dfrac{1+1+0}{-1} = -2$

(f) $\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right)$

Form: $\infty - \infty$. This is a little tricky, since it is not a fractional form. The easiest way to transform this into a form where we can understand the limit is to rationalize the numerator of this expression:

$\dfrac{\sqrt{x^2+1}-x}{1} \cdot \dfrac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x} = \dfrac{x^2+1-x^2}{\sqrt{x^2+1}+x} = \dfrac{1}{\sqrt{x^2+1}+x}$

Therefore, $\lim_{x\to\infty} \left(\sqrt{x^2+1} - x\right) = \lim_{x\to\infty} \dfrac{1}{\sqrt{x^2+1}+x} = 0$

(g) $\lim_{x\to 0} \dfrac{\sin x}{\cos x}$

Form: $\frac{0}{1}$. Thus $\lim_{x\to 0} \dfrac{\sin x}{\cos x} = 0.$

(h) $\lim_{x\to\infty} \left(\dfrac{1}{x}\right)^{\frac{1}{x}}$

Form: 0^0, so we set $y = \left(\frac{1}{x}\right)^{\frac{1}{x}}$, so $\ln y = \frac{1}{x}\ln\left(\frac{1}{x}\right)$.

Form: $0\cdot\infty$, so we change the form into: $\frac{\ln\left(\frac{1}{x}\right)}{x}$, which has the form $\frac{\infty}{\infty}$, so applying L'Hôpital's Rule:

$\lim_{x\to\infty} \dfrac{\ln\left(\frac{1}{x}\right)}{x} = \lim_{x\to\infty} \dfrac{\frac{1}{\frac{1}{x}}\cdot\frac{-1}{x^2}}{1} = \lim_{x\to\infty} \dfrac{-x}{x^2} = \lim_{x\to\infty} \dfrac{-1}{x} = 0.$

Hence $\lim_{x\to\infty} \left(\dfrac{1}{x}\right)^{\frac{1}{x}} = e^0 = 1$

(i) $\lim_{x\to 0^+} (\cos x)^{\frac{1}{x}}$

Form: 1^∞, so we set $y = (\cos x)^{\frac{1}{x}}$, so $\ln y = \frac{1}{x}\ln(\cos x)$.

Form: $\infty\cdot 0$, so we change the form into: $\frac{\ln(\cos x)}{x}$, which has the form $\frac{0}{0}$, so applying L'Hôpital's Rule:

$\lim_{x\to\infty} \dfrac{\ln(\cos x)}{x} = \lim_{x\to\infty} \dfrac{\frac{1}{\cos x}\cdot(-\sin x)}{1} = \lim_{x\to\infty} \dfrac{-\sin x}{\cos x} = 0.$

Hence $\lim_{x\to 0^+} (\cos x)^{\frac{1}{x}} = e^0 = 1$

12. Compute each of the following *exactly*.

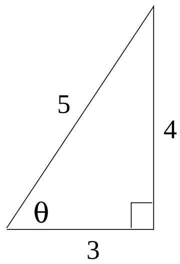
(a) $\arctan(-1) = -\frac{\pi}{4}$

(b) $\arcsin(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$

(c) $\arccos -\frac{\sqrt{2}}{2} = \frac{3\pi}{4}$

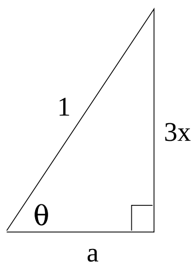
13. Simplify each of the following expressions:

(a) $\tan(\cos^{-1}(\frac{3}{5}))$



Using the Pythagorean Theorem, $5^2 = 3^2 + y^2$, or $25 - 9 = y^2$. Therefore, $y^2 = 16$ or $y = 2$.
Hence $\tan(\cos^{-1}(\frac{3}{5})) = \frac{4}{3}$.

(b) $\cot(\sin^{-1}(3x))$



Similarly, $(3x)^2 + a^2 = 1$, or $a^2 = 1 - 9x^2$. Hence $a = \sqrt{1 - 9x^2}$.
Therefore, $\cot(\sin^{-1}(3x)) = \frac{a}{3x} = \frac{\sqrt{1-9x^2}}{3x}$.

14. Find all solutions to the equation $10\cos^2 x - \cos x = 3$ on the interval $[0, 2\pi)$

Notice that this equation has quadratic form. Therefore,, if we substitute $u = \cos x$, we have $10u^2 - u = 3$. Rearranging this, $10u^2 - u - 3 = 0$, or, factoring, $(2u+1)(5u-3) = 0$. This either $2u + 1 = 0$ or $5u - 3 = 0$
Then either $\cos x = -\frac{1}{2}$, in which case $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$,
or $\cos x = \frac{3}{5}$, in which case, $x \approx .9273$ or $x \approx 2\pi - .9273 \approx 5.3559$

15. Evaluate the following integrals:

(a) $\displaystyle\int \frac{3x^2}{1+x^6}\,dx$

Let $u = x^3$. Then $du = 3x^2\,dx$, and $\displaystyle\int \frac{3x^2}{1+x^6}\,dx = \int \frac{1}{1+u^2}\,du = \arctan u + C = \arctan(x^3) + C$

(b) $\displaystyle\int \frac{4x}{\sqrt{1-x^4}}\,dx$

Let $u = x^2$. Then $du = 2x\,dx$ and $\displaystyle\int \frac{4x}{\sqrt{1-x^4}}\,dx = 2\int \frac{1}{\sqrt{1-u^2}}\,du = 2\arcsin(u) + C = 2\arcsin(x^2) + C$

(c) $\displaystyle\int_{\sqrt{2}}^{2} \frac{4}{x\sqrt{x^2-1}}\,dx$

$$= 4\sec^{-1}(x)\Big|_{\sqrt{2}}^{2} = 4[\sec^{-1}(2) - \sec^{-1}(\sqrt{2})] = 4[\cos^{-1}(\tfrac{1}{2}) - \cos^{-1}(\tfrac{\sqrt{2}}{2})] = 4(\tfrac{\pi}{3} - \tfrac{\pi}{4}) = 4(\tfrac{\pi}{12}) = \tfrac{\pi}{3}$$

(d) $\displaystyle\int_3^4 x\sqrt{x-3}\,dx$

Let $u = x - 3$. Then $du = dx$, and $u + 3 = x$. Therefore, $\displaystyle\int_3^4 x\sqrt{x-3}\,dx = \int_0^1 (u+3)u^{\frac{1}{2}}\,du$

$$= \int_0^1 u^{\frac{3}{2}} + 3u^{\frac{1}{2}}\,du = \frac{2}{5}u^{\frac{5}{2}} + (3)\frac{2}{3}u^{\frac{3}{2}}\Big|_0^1 = \left[(\frac{2}{5} + 2\right) - (0)] = \frac{12}{5}$$

16. Evaluate the following integrals:

(a) $\displaystyle\int \frac{1}{1+x^2}\, dx = \arctan(x) + C$

(b) $\displaystyle\int \frac{x}{1+x^2}\, dx$

Let $u = 1 + x^2$. Then $du = 2x\,dx$, or $\frac{1}{2}du = x\,dx$.

Therefore, $\displaystyle\int \frac{x}{1+x^2}\, dx = \frac{1}{2}\int \frac{1}{u}\, du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C = \frac{1}{2}\ln(x^2+1) + C$

(c) $\displaystyle\int \frac{x^2}{1+x^2}\, dx = \int \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\, dx = \int 1 - \frac{1}{1+x^2}\, dx$

$= x - \arctan(x) + C$

(d) $\displaystyle\int \frac{x^3}{1+x^2}\, dx$

Let $u = x^2 + 1$. Then $du = 2x\,dx$ or $\frac{1}{2}du = x\,dx$ and $u - 1 = x^2$.

Therefore, $\displaystyle\int \frac{x^3}{1+x^2}\, dx = \int \frac{x^2 \cdot x}{1+x^2}\, dx = \frac{1}{2}\int \frac{(u-1))}{u}\, du = \frac{1}{2}\int 1 - \frac{1}{u}\, du$

$= \frac{1}{2}[u - \ln|u|] + C = \frac{x^2+1}{2} - \frac{1}{2}\ln(x^2+1) + C$