Math 262
Exam 4 - Practice Problem Solutions

1. For each of the following sequences, determine whether the sequence converges or diverges. If a sequence converges, whenever possible, find the value of the limit of the sequence.

   (a) $\left\{\dfrac{n+2}{3n-1}\right\}$

   Notice that $\lim\limits_{x\to\infty} \dfrac{x+2}{3x-1} = \dfrac{1}{3}$. Therefore, this sequence converges to $\dfrac{1}{3}$.

   (b) $\left\{(-1)^n \dfrac{n+2}{3n-1}\right\}$

   Notice that if we consider the absolute value of this sequence: $\lim\limits_{x\to\infty} \dfrac{x+2}{3x-1} = \dfrac{1}{3}$.
   From this, we see that the subsequence of even terms of this sequence converges to $\frac{1}{3}$ while the subsequence of odd terms converges to $-\frac{1}{3}$. Hence this sequence diverges.

   (c) $\{ne^{-n}\}$

   Notice that $\lim\limits_{x\to\infty} xe^{-x} = \lim\limits_{x\to\infty} \dfrac{x}{e^x} = \lim\limits_{x\to\infty} \dfrac{1}{e^x} = 0$.
   Therefore, this sequence converges to 0.

   (d) $\left\{\dfrac{\cos n}{e^n}\right\}$

   Notice that $\dfrac{-1}{e^n} \le \dfrac{\cos n}{e^n} \le \dfrac{1}{e^n}$
   Also, $\lim\limits_{x\to\infty} \dfrac{-1}{e^n} = 0$ and $\lim\limits_{x\to\infty} \dfrac{1}{e^n} = 0$, so by the sandwich theorem for sequences, $\lim\limits_{x\to\infty} \dfrac{\cos n}{e^n} = 0$

   (e) $\{\sqrt[n]{n}\}$

   Consider the limit of the related function: $\lim\limits_{x\to\infty} \sqrt[x]{x} = \lim\limits_{x\to\infty} x^{\frac{1}{x}}$.
   Taking the natural logarithm of this gives: $\lim\limits_{x\to\infty} \dfrac{1}{x}\ln x = \lim\limits_{x\to\infty} \dfrac{\ln x}{x}$ which, by L'Hôpital's Rule:
   $= \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{1} = \lim\limits_{x\to\infty} \dfrac{1}{x} = 0$.
   Then $\lim\limits_{x\to\infty} x^{\frac{1}{x}} = e^0 = 1$. Hence $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

   (f) $\left\{\dfrac{n2^n}{3^n}\right\}$

   First, notice that $a_{n+1} = \dfrac{(n+1)2^{n+1}}{3^{n+1}} = (n+1)\dfrac{2}{3}\cdot\left(\dfrac{2}{3}\right)^n$.
   Also, when $n > 2$, $2n+2 < 3n$, so $\dfrac{2n+2}{3} < n$ or $0 < (n+1)\frac{2}{3} < n$. Hence for $n > 2$, $a_n > a_{n+1} \ge 0$.
   But this means that this sequence is both monotone and bounded. Hence this sequence converges.

   (g) $\left\{\left(1+\dfrac{2}{n}\right)^{2n}\right\}$

   Again making use of logarithms and L'Hôpital's Rule:
   $\lim\limits_{x\to\infty} 2x\ln\left(1+\dfrac{2}{x}\right) = 2\lim\limits_{x\to\infty} \dfrac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} = 2\lim\limits_{x\to\infty} \dfrac{\frac{1}{1+\frac{2}{x}}\cdot\left(-2x^{-2}\right)}{-x^{-2}}$
   $= 4\lim\limits_{x\to\infty} \dfrac{1}{1+\frac{2}{x}} = 4$
   Hence $\lim\limits_{n\to\infty} \left(1+\dfrac{2}{n}\right)^{2n} = e^4$

2. Suppose $a_1 = 1$ and $a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{4}{a_n}\right)$

(a) Compute a_5

$a_1 = 1$; $a_2 = \frac{1}{2}\left(1 + \frac{4}{1}\right) = \frac{5}{2}$; $a_3 = \frac{1}{2}\left(\frac{5}{2} + \frac{4\cdot 2}{5}\right) = \frac{1}{2}\left(\frac{5}{2} + \frac{8}{5}\right) = \frac{41}{20}$

$a_4 = \frac{1}{2}\left(\frac{41}{20} + \frac{4\cdot 20}{41}\right) = \frac{1}{2}\left(\frac{41^2+4\cdot 20^2}{41\cdot 20}\right) = \frac{3281}{1640}$

$a_5 = \frac{1}{2}\left(\frac{3281}{1640} + \frac{4\cdot 1640}{3281}\right) = \frac{1}{2}\left(\frac{3281^2+4\cdot 1640^2}{1640\cdot 3281}\right) = \frac{21523361}{10761680}$

(b) Find $\lim_{n\to\infty} a_n$ [Hint: Let $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$. Then $L = \dfrac{1}{2}\left(L + \dfrac{4}{L}\right)$]

Solving $L = \dfrac{1}{2}\left(L + \dfrac{4}{L}\right)$ for L:

$2L = \dfrac{L^2+4}{L}$, so $2L^2 = L^2 + 4$. Thus $L^2 = 4$, hence $L = \pm 2$.

Since a_1 is positive, and whenever a_n is positive, so is a_{n+1}, we can reject the negative solution and conclude that $L = 2$.

3. Determine whether the following series converge or diverge. For those that converge, find the sum of the series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2}\left(-\frac{1}{3}\right)^n$

This is a geometric series with $a = -\frac{1}{6}$ and $r = -\frac{1}{3}$. Clearly, $|r| < 1$. Therefore, $S = \dfrac{a}{1-r} = \dfrac{-\frac{1}{6}}{1-\left(-\frac{1}{3}\right)} = \dfrac{-\frac{1}{6}}{\frac{4}{3}} = -\dfrac{1}{6}\cdot\dfrac{3}{4} = -\dfrac{1}{8}$.

(b) $\displaystyle\sum_{n=1}^{\infty} 4\left(\frac{1}{2}\right)^n$

This is a geometric series with $a = 2$ and $r = \frac{1}{2}$. Clearly, $|r| < 1$. Therefore, $S = \dfrac{a}{1-r} = \dfrac{2}{1-\left(\frac{1}{2}\right)} = \dfrac{2}{\frac{1}{2}} = 4$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{4n}{n+2}$

Notice that $\displaystyle\lim_{n\to\infty} \frac{4n}{n+2} = \lim_{n\to\infty} \frac{4}{1} = 4$. Therefore, this series diverges by the nth term test.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{9}{n(n+3)}$

Using partial fractions, we can rewrite $\dfrac{9}{n(n+3)} = \dfrac{A}{n} + \dfrac{B}{n+3}$, where $A(n+3) + Bn = 9$.

Setting $n = 0$ gives $3A = 9$ or $A = 3$. Setting $n = -3$ gives $-3B = 9$ or $B = -3$.

Then we have $\dfrac{9}{n(n+3)} = \dfrac{3}{n} - \dfrac{3}{n+3} = 3\left(\dfrac{1}{n} - \dfrac{1}{n+3}\right)$

Therefore, this is a telescoping series of the form: $3\left(1 - \dfrac{1}{4} + \dfrac{1}{2} - \dfrac{1}{5} + \dfrac{1}{3} - \dfrac{1}{6}...\right)$

Hence for $n \geq 3$, $S_n = 3\left(1 + \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{n+1} - \dfrac{1}{n+2} - \dfrac{1}{n+3}\right)$

Thus $\displaystyle\lim_{n\to\infty} S_n = 3\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{2}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$

Using partial fractions, we can rewrite $\dfrac{4}{n(n+2)} = \dfrac{A}{n} + \dfrac{B}{n+2}$, where $A(n+2) + Bn = 4$.

Setting $n = 0$ gives $2A = 4$ or $A = 2$. Setting $n = -2$ gives $-2B = 4$ or $B = -2$.

Then we have $\dfrac{4}{n(n+2)} = \dfrac{2}{n} - \dfrac{2}{n+2} = 2\left(\dfrac{1}{n} - \dfrac{1}{n+2}\right)$

Therefore, this is a telescoping series of the form: $2\left(1 - \dfrac{1}{3} + \dfrac{1}{2} - \dfrac{1}{4} + \dfrac{1}{3} - \dfrac{1}{5}...\right)$

Hence for $n \geq 3$, $S_n = 2\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$

Thus $\lim_{n \to \infty} S_n = 2\left(1 + \frac{1}{2}\right) = 3$

(f) $\sum_{n=1}^{\infty} (-1)^n \frac{4}{3^n}$

Notice that $\sum_{n=1}^{\infty} (-1)^n \frac{4}{3^n} \sum_{n=1}^{\infty} 4\left(\frac{-1}{3}\right)^n$

This is a geometric series with $a = -\frac{4}{3}$ and $r = -\frac{1}{3}$. Clearly, $|r| < 1$. Therefore, $S = \dfrac{a}{1-r} = \dfrac{-\frac{4}{3}}{1-\left(-\frac{1}{3}\right)} = \dfrac{-\frac{4}{3}}{\frac{4}{3}} = -1$.

4. Use geometric series to express each of the following repeating decimals in fractional form.

(a) $.11\overline{1}$

Notice that this repeating decimal can be written as the series: $\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$

This is a geometric series with $a = \frac{1}{10}$ and $r = \frac{1}{10}$. Clearly, $|r| < 1$. Therefore, $S = \dfrac{a}{1-r} = \dfrac{\frac{1}{10}}{1-\left(\frac{1}{10}\right)} = \dfrac{\frac{1}{10}}{\frac{9}{10}} = \dfrac{1}{9}$.

(b) $.7878\overline{78}$

Notice that this repeating decimal can be written as the series: $\sum_{n=1}^{\infty} 78\left(\frac{1}{100}\right)^n$

This is a geometric series with $a = \frac{78}{100}$ and $r = \frac{1}{100}$. Clearly, $|r| < 1$. Therefore, $S = \dfrac{a}{1-r} = \dfrac{\frac{78}{100}}{1-\left(\frac{1}{100}\right)} = \dfrac{\frac{78}{100}}{\frac{99}{100}} = \dfrac{78}{99}$.

(c) $.137137\overline{137}$

Notice that this repeating decimal can be written as the series: $\sum_{n=1}^{\infty} 137\left(\frac{1}{1000}\right)^n$

This is a geometric series with $a = \frac{137}{1000}$ and $r = \frac{1}{1000}$. Clearly, $|r| < 1$. Therefore, $S = \dfrac{a}{1-r} = \dfrac{\frac{137}{100}}{1-\left(\frac{1}{1000}\right)} = \dfrac{\frac{137}{1000}}{\frac{999}{1000}} = \dfrac{137}{999}$.

(d) $.99\overline{9}$

Notice that this repeating decimal can be written as the series: $\sum_{n=1}^{\infty} 9\left(\frac{1}{10}\right)^n$

This is a geometric series with $a = \frac{9}{10}$ and $r = \frac{1}{10}$. Clearly, $|r| < 1$. Therefore, $S = \dfrac{a}{1-r} = \dfrac{\frac{9}{10}}{1-\left(\frac{1}{10}\right)} = \dfrac{\frac{9}{10}}{\frac{9}{10}} = 1$.

5. For each of the following series, if the series is positive term, determine whether it is convergent or divergent; if the series contains negative terms, determine whether it is absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{4}{n(\ln n)^3}$

Notice that $f(x) = \dfrac{4}{x(\ln x)^3}$ is continuous and decreasing for $x \geq 2$.

Consider $\displaystyle\int_2^{\infty} \frac{4}{x(\ln x)^3}\,dx$. If we let $u = \ln x$, then $du = \frac{1}{x}\,dx$. Then, rewriting this as an improper integral:

$$\lim_{t\to\infty} \int_{\ln 2}^{\ln t} 4u^{-3}\,du = \lim_{t\to\infty} -2u^{-2}\Big|_{\ln 2}^{\ln t} = \lim_{t\to\infty} -\frac{2}{(\ln t)^2} + \frac{2}{(\ln 2)^2}$$

which converges. Therefore, the series $\displaystyle\sum_{n=2}^{\infty} \frac{4}{n(\ln n)^3}$ converges by the integral test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{1+n^{-1}}}{n^2}$

Since $\dfrac{1}{n} \leq 1$ for $n \geq 1$, $\dfrac{\sqrt{1+n^{-1}}}{n^2} = \dfrac{\sqrt{1+\frac{1}{n}}}{n^2} \leq \dfrac{\sqrt{2}}{n^2}$. Also, $\sum_{n=1}^{\infty} \frac{\sqrt{2}}{n^2}$ is a convergent p-series. Thus the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{1+n^{-1}}}{n^2}$ converges by comparison.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n - 2}{n^2}$

Notice that $\left|\dfrac{\sin n - 2}{n^2}\right| \leq \dfrac{3}{n^2}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2}$ is a convergent p-series, the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n - 2}{n^2}$ converges by comparison.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4 + 2n - 1}{n^5 + 3n^2 - 20}$

Using the limit comparison test, let $b_n = \dfrac{1}{n}$.

Then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^4 + 2n - 1}{n^5 + 3n^2 - 20} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^5 + 2n^2 - n}{n^5 + 3n^2 - 20} = 1.$

Therefore, since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, $\displaystyle\sum_{n=1}^{\infty} \frac{n^4 + 2n - 1}{n^5 + 3n^2 - 20}$ diverges by the Limit Comparison Test.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{\left(\frac{1}{n}+1\right)}}{n^3}$

Since $\dfrac{1}{n} \leq 1$ for $n \geq 1$, $e^{\left(\frac{1}{n}+1\right)} \leq e^2$ Therefore, $\dfrac{e^{\left(\frac{1}{n}+1\right)}}{n^3} \leq \dfrac{e^2}{n^3}$.

But $\displaystyle\sum_{n=1}^{\infty} \frac{e^2}{n^3} = e^2 \sum_{n=1}^{\infty} \frac{1}{n^3}$, which is a convergent p-series. Thus $\displaystyle\sum_{n=1}^{\infty} \frac{e^{\left(\frac{1}{n}+1\right)}}{n^3}$ converges by comparison.

(f) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{4}{n+1}$

First notice that $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n+1}$ diverges, since if we let $b_n = \dfrac{1}{n}$.

Then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{4}{n+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{4n}{n+1} = 4.$

Therefore, since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n+1}$ diverges by the Limit Comparison Test. Hence $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{4}{n+1}$ is **not** absolutely convergent.

Next, notice that $\displaystyle\lim_{n\to\infty} \frac{4}{n+1} = 0$, and $\dfrac{4}{n+1} \geq \dfrac{4}{n+2}$. Thus by the Alternating Series test, $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{4}{n+1}$ is conditionally convergent.

(g) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4n}{5n+1}\right)^n$

Using the Root Test, notice that $\displaystyle\sqrt[n]{\left(\frac{4n}{5n+1}\right)^n} = \frac{4n}{5n+1}$. Moreover, $\displaystyle\lim_{n\to\infty}\frac{4n}{5n+1} = \frac{4}{5} < 1$

Hence, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4n}{5n+1}\right)^n$ converges by the Root Test.

(h) $\displaystyle\sum_{n=1}^{\infty}\frac{2\cdot n}{3^n}$

Using the Ratio Test, $\displaystyle a_{n+1} = \frac{2(n+1)}{3^{n+1}}$.

Then $\displaystyle\frac{a_{n+1}}{a_n} = \frac{2(n+1)}{3^{n+1}}\cdot\frac{3^n}{2n} = \frac{n+1}{3n}$.

Therefore, $\displaystyle\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{n+1}{3n} = \frac{1}{3} < 1$. Hence $\displaystyle\sum_{n=1}^{\infty}\frac{2\cdot n}{3^n}$ converges by the Ratio Test.

(i) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{4^n}{(2n+1)!}$

We first check for absolute convergence by applying the ratio test to $\displaystyle\sum_{n=1}^{\infty}\frac{4^n}{(2n+1)!}$:

Notice that $\displaystyle a_{n+1} = \frac{4^{n+1}}{(2(n+1)+1)!} = \frac{4^{n+1}}{(2n+3)!}$.

Then $\displaystyle\frac{a_{n+1}}{a_n} = \frac{4^{n+1}}{(2n+3)!}\cdot\frac{(2n+1)!}{4^n} = \frac{4}{(2n+3)(2n+2)}$.

Therefore, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \displaystyle\lim_{n\to\infty}\frac{4}{(2n+3)(2n+2)} = 0 < 1$. Hence $\displaystyle\sum_{n=1}^{\infty}\frac{4^n}{(2n+1)!}$ converges by the Ratio Test.

Thus $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{4^n}{(2n+1)!}$ converges absolutely.

(j) $\displaystyle\sum_{n=1}^{\infty}n^3e^{-n}$

Using the Ratio Test, $\displaystyle a_n = \frac{n^3}{e^n}$ and $\displaystyle a_{n+1} = \frac{(n+1)^3}{e^{n+1}} = \frac{n^3+3n^2+3n+1}{e^{n+1}}$.

Then $\displaystyle\frac{a_{n+1}}{a_n} = \frac{n^3+3n^2+3n+1}{e^{n+1}}\cdot\frac{e^n}{n^3} = \frac{n^3+3n^2+3n+1}{en^3}$.

Therefore, $\displaystyle\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{n^3+3n^2+3n+1}{en^3} = \frac{1}{e} < 1$. Hence $\displaystyle\sum_{n=1}^{\infty}n^3e^{-n}$ converges by the Ratio Test.

(k) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

First notice that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a divergent p-series ($p = \frac{1}{2} \le 1$),so this series does not converge absolutely.

Next, $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$ and $\frac{1}{\sqrt{n+1}} \le \frac{1}{\sqrt{n}}$.

Hence, by the Alternating Series Test, this series converges conditionally.

(l) $\displaystyle\sum_{n=1}^{\infty}\frac{4^n}{(n!)^2}$

Using the Ratio Test, $\displaystyle a_{n+1} = \frac{4^{n+1}}{((n+1)!)^2} = \frac{4^{n+1}}{(n+1)!(n+1)!}$.

Then $\displaystyle\frac{a_{n+1}}{a_n} = \frac{4^{n+1}}{(n+1)!(n+1)!}\cdot\frac{n!n!}{4^n} = \frac{4}{(n+1)^2}$.

Therefore, $\displaystyle\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{4}{(n+1)^2} = 0 < 1$. Hence $\displaystyle\sum_{n=1}^{\infty}\frac{4^n}{(n!)^2}$ converges by the Ratio Test.

(m) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt[n]{n}}$

Recall that we can compute $\lim_{x\to\infty} \sqrt[n]{n}$ as follows:

Consider the limit of the related function: $\lim_{x\to\infty} \sqrt[x]{x} = \lim_{x\to\infty} x^{\frac{1}{x}}$.

Taking the natural logarithm of this gives: $\lim_{x\to\infty} \frac{1}{x} \ln x = \lim_{x\to\infty} \frac{\ln x}{x}$ which, by L'Hôpital's Rule:

$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$

Then $\lim_{x\to\infty} x^{\frac{1}{x}} = e^0 = 1$. Hence $\lim_{n\to\infty} \sqrt[n]{n} = 1$

But then $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1$, and hence $\lim_{n\to\infty} (-1)^n \frac{1}{\sqrt[n]{n}}$ does not exist.

Thus, be this series diverges by the nth term test.

6. Estimate the sum of the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^4+1}$ to within 0.01

First notice that if $f(x) = \frac{x}{x^4+1}$, then $f'(x) = \frac{(x^4+1) - x(4x^3)}{(x^4+1)^2} = \frac{-3x^4+1}{(x^4+1)^2} < 0$ whenever $x \geq 1$.

Next, $\lim_{n\to\infty} \frac{n}{n^4+1} = 0$. Then, by the Error Estimation Theorem for Alternating Series, we need to find n such that $a_{n+1} < 0.01$.

Since I really don't feel like solving a 4th degree polynomial equation that does not factor, we'll find n by brute force. Notice that $a_4 = \frac{4}{4^4+1} = \frac{4}{257} \approx 0.015564$ while $a_5 = \frac{5}{5^4+1} \approx 0.007987$

Therefore, we can apporximate S to within 0.01 by adding the first 4 terms of this series:

$S_4 = -\frac{1}{2} + \frac{2}{17} - \frac{3}{82} + \frac{4}{257} \approx -0.40$

7. Determine the number of terms necessary to estimate the sum of the following series to within 1×10^{-6}

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{3}{n^2}$

Notice that this series is decreasing and its terms tend to 0 as $n \to \infty$

If $\frac{3}{n^2} < 10^{-6}$, then $\frac{3}{10^{-6}} < n^2$, so $n^2 > \sqrt{\frac{3}{10^{-6}}} = \sqrt{3000000} \approx 1732.05$, so we can estimate S to within 10^{-6} by computing S_n with $n = 1732$.

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}$

Notice that this series is decreasing and its terms tend to 0 as $n \to \infty$

Since the algebra is quite challenging, we will find n by brute force:

Notice that $a_{10} = \frac{2^{10}}{10!} \approx .000282$; $a_{12} = \frac{2^{12}}{12!} \approx .000008551$

$a_{13} = \frac{2^{13}}{13!} \approx .000001316$; $a_{14} = \frac{2^{14}}{14!} \approx .000000188$

So we can estimate S to within 10^{-6} by computing S_n with $n = 13$.

8. Find all real values of x for which the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n \cdot 4^n}$ converges.

We first use the ratio test on the positive part of this series:

Notice that $a_{n+1} = \frac{x^{n+1}}{(n+1)4^{n+1}} = \frac{4^{n+1}}{(n+1)!(n+1)!}$.

Then $\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)4^{n+1}} \cdot \frac{n4^n}{x^n} = \frac{nx}{4(n+1)}$.

Therefore, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x}{4} \cdot \frac{n}{n+1} = \frac{x}{4}$. Hence, by the Ratio Test, $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n \cdot 4^n}$ converges absolutely when $x < 4$ and diverges when $x > 4$.

This test is inconclusive when $|x| = 4$.

When $x = 4$, we have $\sum_{n=1}^{\infty} (-1)^n \frac{4^n}{n \cdot 4^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, which converges conditionally by the alternating series test (the positive part of this series is clearly decreasing and the terms tend to zero).

When $x = -4$, we have $\sum_{n=1}^{\infty} (-1)^n \frac{(-4)^n}{n \cdot 4^n} = \sum_{n=1}^{\infty} (-1)^{2n} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

Therefore, this series converges for all x-values in the interval $(-4.4]$.

9. For each of the following power series, find the interval of convergence and the radius of convergence:

(a) $\sum_{n=1}^{\infty} (-1)^n n^2 x^n$

Notice that $a_{n+1} = (-1)^{n+1}(n+1)^2 x^{n+1}$. Then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2|x|^{n+1}}{n^2|x|^n} = \lim_{n\to\infty} |x| \frac{n^2+2n+1}{n^2}$
$= |x| \lim_{n\to\infty} \frac{2n+2}{2n} = |x| \lim_{n\to\infty} \frac{2}{2} = |x|$, so this series converges absolutely for $-1 < x < 1$.

Notice when $x = 1$, we have $\sum_{n=1}^{\infty} (-1)^n n^2 1^n = \sum_{n=1}^{\infty} (-1)^n n^2$ which diverges by the nth term test.

Similarly, when $x = -1$, we have $\sum_{n=1}^{\infty} (-1)^n n^2 (-1)^n = \sum_{n=1}^{\infty} (-1)^2 n n^2 = \sum_{n=1}^{\infty} 1$ which diverges by the nth term test.

Hence, the interval of convergence is: $(-1, 1)$ and the radius convergence is: $R = 1$.

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}(x-3)^n$

Notice that $a_{n+1} = \frac{2^{n+1}}{(n+1)^2}(x-3)^{n+1}$. Then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}|x-3|^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n|x-3|^n}$
$= \lim_{n\to\infty} |x-3| \cdot 2 \cdot \frac{n^2+2n+1}{n^2} = 2|x-3| \lim_{n\to\infty} \frac{2n+2}{2n} = 2|x-3| \lim_{n\to\infty} \frac{2}{2} = 2|x-3|$, so this series converges absolutely when $|x-3| < \frac{1}{2}$, or for $\frac{5}{2} < x < \frac{7}{2}$.

Notice when $x = \frac{5}{2}$, we have $\sum_{n=1}^{\infty} \frac{2^n}{n^2}(-\frac{1}{2})^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ Thus, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, the original series converges absolutely.

Similarly, when $x = \frac{7}{2}$, we have $\sum_{n=1}^{\infty} \frac{2^n}{n^2}(\frac{1}{2})^n = \sum_{n=1}^{\infty} \frac{(1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series.

Hence, the interval of convergence is: $\left[\frac{5}{2}, \frac{7}{2}\right]$ and the radius convergence is: $R = \frac{1}{2}$.

(c) $\sum_{n=1}^{\infty} \frac{n^3}{3^n}(x+1)^n$

Notice that $a_{n+1} = \frac{(n+1)^3}{3^{n+1}}(x+1)^{n+1}$. Then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^3|x+1|^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n^3|x+1|^n}$
$= \frac{1}{3}|x+1| \lim_{n\to\infty} \frac{(n+1)^3}{n^3}$, which, after a few applications of L'Hôpital's Rule, is $\frac{|x+1|}{3}$, so this series converges absolutely when $|x+1| < 3$ or for $-4 < x < 2$.

Notice when $x = -4$, we have $\sum_{n=1}^{\infty} \frac{n^3}{3^n}(-3)^n = \sum_{n=1}^{\infty} (-1)^n n^3$, which diverges by the nth term test.

Similarly, when $x = 2$, we have $\sum_{n=1}^{\infty} \frac{n^3}{3^n} 3^n = \sum_{n=1}^{\infty} n^3$ which diverges by the nth term test.

Hence, the interval of convergence is: $(-4, 2)$ and the radius convergence is: $R = 3$.

(d) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{10^n}{n!}(x-10)^n$

Notice that $a_{n+1}=(-1)^{n+1}\dfrac{10^{n+1}}{(n+1)!}(x-10)^{n+1}$. Then $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{10^{n+1}|x-10|^{n+1}}{(n+1)!}\cdot\frac{n!}{10^n|x-10|^n}$

$=|x-10|\displaystyle\lim_{n\to\infty}\frac{10}{n+1}=0$

Hence the interval of convergence is $(-\infty,\infty)$ and $R=\infty$.

(e) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n10^n}(x-2)^n$

Notice that $a_{n+1}=(-1)^{n+1}\dfrac{1}{(n+1)10^{n+1}}(x-2)^{n+1}$. Then $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{|x-2|^{n+1}}{(n+1)10^{n+1}}\cdot\frac{n10^n}{|x-2|^n}$

$=\dfrac{1}{10}|x-2|\displaystyle\lim_{n\to\infty}\frac{n}{n+1}=\frac{1}{10}|x-2|$, so this series converges absolutely when $|x-2|<10$ or for $-8<x<12$.

Notice when $x=-8$, we have $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n10^n}(-10)^n=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}(-1)^n=\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges since it is the harmonic series.

Similarly, when $x=10$, we have $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n10^n}10^n=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ which converges by the Alternating Series Test.

Hence, the interval of convergence is: $(-8,10]$ and the radius convergence is: $R=10$.

10. Use a known series to find a power series in x that has the given function as its sum:

(a) $x\sin(x^3)$

Recall the Maclaurin series for $\sin u=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{u^{2n+1}}{(2n+1)!}$

Therefore, $\sin(x^3)=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(x^3)^{2n+1}}{(2n+1)!}=\sum_{n=0}^{\infty}(-1)^n\frac{(x)^{6n+3}}{(2n+1)!}$.

Hence $x\sin(x^3)=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(x)^{6n+4}}{(2n+1)!}$.

(b) $\dfrac{\ln(1+x)}{x}$

Recall the Maclaurin series for $\ln(1+x)=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$

Therefore, $\dfrac{\ln(1+x)}{x}=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n+1}$

(c) $\dfrac{x-\arctan x}{x^3}$

Recall the Maclaurin series for $\arctan(x)=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$

Therefore, $x-\arctan(x)=x-\left(x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\cdots\right)=\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{2n+1}$

Hence $\dfrac{x-\arctan x}{x^3}=\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-2}}{2n+1}$