Math 262
Exam 5 - Practice Problems

1. Use a power series to approximate each of the following to within 3 decimal places:

   (a) $\arctan \frac{1}{2}$

   Notice that the Maclaurin series $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ is an alternating series satisfying the hypotheses of the alternating series test when $x = \frac{1}{2}$. Then to find our approximation, we need to find n such that $\frac{(.5)^{2n+1}}{2n+1} < .0005$.
   $a_0 = \frac{1}{2}$, $a_1 = -\frac{1}{24} \approx 0.04667$, $a_3 = \frac{1}{160} = 0.00625$, $a_4 = -\frac{1}{896} \approx -0.001116$, and $a_5 \approx 0.00217$
   Hence $\arctan \frac{1}{2} \approx \frac{1}{2} - \frac{1}{24} + \frac{1}{160} - \frac{1}{896} \approx 0.463$

   (b) $\ln(1.01)$

   Notice that the Maclaurin series $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$ is an alternating series satisfying the hypotheses of the alternating series test when $x = 0.01$. Then to find our approximation, we need to find n such that $\frac{(0.1)^{n+1}}{n+1} < .0005$.
   $a_0 = 0.01$, $a_1 = -0.00005$
   Hence $\ln(1.01) \approx 0.010$

   (c) $\sin\left(\frac{\pi}{10}\right)$

   Notice that the Maclaurin series $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ is an alternating series satisfying the hypotheses of the alternating series test when $x = \frac{\pi}{10}$. Then to find our approximation, we need to find n such that $\frac{(\frac{\pi}{10})^{2n+1}}{(2n+1)!} < .0005$.
   $a_0 = \frac{\pi}{10} \approx 0.314159$, $a_1 \approx -0.0051677$, $a_2 \approx 0.0000255$
   Hence $\sin\left(\frac{\pi}{10}\right) \approx 0.314159 - 0.0051677 \approx 0.309$

2. For each of the following functions, find the Taylor Series about the indicated center and also determine the interval of convergence for the series.

   (a) $f(x) = e^{x-1}$, $c = 1$

   Notice that $f'(x) = e^{x-1}$ and $f''(x) = e^{x-1}$. In fact, $f^{(n)}(x) = e^{x-1}$ for every n.
   Then $f^{(n)}(1) = e^0 = 1$ for every n, and hence $a_n = \frac{1}{n!}$ for every n.
   Thus $e^{x-1} = \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$.

   To find the interval of convergence, notice that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-1|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x-1|^n} = |x-1| \lim_{n\to\infty} \frac{1}{n+1} = 0$
   Thus this series converges on $(-\infty, \infty)$ and $R = \infty$.

(b) $f(x) = \cos x$, $c = \dfrac{\pi}{2}$

$f'(x) = -\sin x$, $f''(x) = \cos x$, $f'''(x) = \sin x$, $f^4(x) = -\cos x$, and the same pattern continues from there.
Therefore, $f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$ $f'\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$, $f''\left(\frac{\pi}{2}\right) = -\cos\frac{\pi}{2} = 0$, $f'''\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$, $f^4\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$, and the pattern continues from there.
Therefore, $a_0 = 0$, $a_1 = -1$, $a_2 = 0$, $a_3 = \frac{1}{3!} = \frac{1}{6} \cdots$
Hence the series is: $\cos x = \sum_{n=0}^{\infty}(-1)^{n+1}\dfrac{1}{(2n+1)!}(x-\frac{\pi}{2})^{2n+1}$

To find the interval of convergence, notice that $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\dfrac{|x-\frac{\pi}{2}|^{n+1}}{(2n+3)!}\cdot\dfrac{(2n+1)!}{|x-\frac{\pi}{2}|^n}$
$= |x-\dfrac{\pi}{2}|\lim_{n\to\infty}\dfrac{1}{(2n+3)(2n+2)} = 0$
Thus this series converges on $(-\infty, \infty)$ and $R = \infty$.

(c) $f(x) = \dfrac{1}{x}$, $c = -1$

$f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$, so $f^n(x) = (-1)^n x^{-(n+1)}$
Then $f(-1) = -1$, $f'(-1) = -1$, $f''(-1) = -2$, $f'''(-1) = -6$, and $f^n(-1) = -n!$.
Therefore, $a_0 = -1$, $a_1 = -1$, $a_2 = -1$, $a_3 = -1$, and, in fact, $a_n = -1$ for all n.
Hence $\dfrac{1}{x} = \sum_{n=0}^{\infty}(-1)(x+1)^n$

To find the interval of convergence, notice that $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\dfrac{(-1)|x-1|^{n+1}}{(-1)|x-1|^n} = |x-1|$, so this series converges absolutely for $0 \le x \le 2$
When $x = 0$, we have $\sum_{n=0}^{\infty}(-1)(-1)^n$, which diverges by the nth term test.
Similarly, when $x = 2$ we have $\sum_{n=0}^{\infty}(-1)(1)^n$, which also diverges by the nth term test.
Thus this series converges on $(0, 2)$ and $R = 1$.

3. For each of the following functions, find the Taylor Polynomial for the function at the indicated center c. Also find the Remainder term.

(a) $f(x) = \sqrt{x}$, $c = 1$, $n = 3$.

First, $f'(x) = \dfrac{1}{2}x^{-\frac{1}{2}}$, $f''(x) = -\dfrac{1}{4}x^{-\frac{3}{2}}$, $f'''(x) = \dfrac{3}{8}x^{-\frac{5}{2}}$, and $f^{(4)}(x) = -\dfrac{15}{16}x^{-\frac{7}{2}}$.
Then $f(1) = 1$, $f'(1) = \dfrac{1}{2}$, $f''(1) = -\dfrac{1}{4}$, $f'''(1) = \dfrac{3}{8}$.
Hence $a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = -\frac{1}{8}$, and $a_3 = \frac{1}{16}$
Thus $P_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$
and $R_3(x) = \frac{f^{(4)}(z)}{4!}(x-1)^4 = \frac{5z^{-\frac{7}{2}}}{128}(x-1)^4$

(b) $f(x) = \ln x$, $c = 1$, $n = 4$.

First, $f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = 2x^{-3}$, $f^{(4)}(x) = -6x^{-4}$, and $f^{(5)}(x) = 24x^{-5}$.
Then $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$, and $f^{(4)}(1) = -6$.
Hence $a_0 = 0$, $a_1 = 1$, $a_2 = -\frac{1}{2}$, $a_3 = \frac{1}{3}$, and $a_4 = -\frac{1}{4}$
Thus $P_4(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$

and $R_4(x) = \frac{f^{(5)}(z)}{5!}(x-1)^5 = \frac{24z^{-5}}{120}(x-1)^5 = \frac{z^{-5}}{5}(x-1)^5$

(c) $f(x) = \sqrt{1+x^2}$, $c = 0$, $n = 4$.

First, $f'(x) = x(1+x^2)^{-\frac{1}{2}}$, $f''(x) = (1+x^2)^{-\frac{1}{2}} - x^2(1+x^2)^{-\frac{3}{2}}$, $f'''(x) = -3x(1+x^2)^{-\frac{3}{2}} + 3x^3(1+x^2)^{-\frac{5}{2}}$, $f^{(4)}(x) = -3(1+x^2)^{-\frac{3}{2}} + 18x^2(1+x^2)^{-\frac{5}{2}} - 15x^4(1+x^2)^{-\frac{7}{2}}$, and $f^{(5)}(x) = 45x(1+x^2)^{-\frac{5}{2}} - 150x^3(1+x^2)^{-\frac{7}{2}} + 105x^5(1+x^2)^{-\frac{9}{2}}$
Then $f(0) = 1$, $f'(0) = 0$, $f''(0) = 1$, $f'''(0) = 0$, and $f^{(4)}(0) = -3$.
Hence $a_0 = 1$, $a_1 = 0$, $a_2 = \frac{1}{2}$, $a_3 = 0$, and $a_4 = -\frac{1}{8}$
Thus $P_4(x) = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4$
and $R_4(x) = \frac{f^{(5)}(z)}{5!}x^5 = \frac{45z(1+z^2)^{-\frac{5}{2}} - 150z^3(1+z^2)^{-\frac{7}{2}} + 105z^5(1+z^2)^{-\frac{9}{2}}}{120}x^5$

4. Estimate each of the following using a Taylor Polynomial of degree 4. Also find the error or your approximation. Finally, find the number of terms needed to guarantee an accuracy or at least 5 decimal places.

(a) $e^{0.1}$

Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.
Then $P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$, and $R_4 = \frac{e^z}{5!}x^5$
When $x = 0.1$, $P_4(x) \approx 1 + 0.1 + 0.005 + 0.0001667 + .000004167 = 1.105170867$
In general, $R_n(x) = \frac{f^{(n+1)(z)}}{(n+1)!}x^{n+1} = \frac{e^z}{(n+1)!}(0.1)^{n+1}$, where $0 \le z \le 0.1$.
Since e^x is increasing, we need to find n so that $\frac{e^{0.1}}{(n+1)!}(0.1)^{n+1} < 0.000005$

When we use $P_4(x)$, our error is at most $\frac{e^{0.1}}{5!}(0.1)^5 \approx 0.000000092$ (in fact, one would only need $P_3(x)$ to get within 5 decimal places).

(b) $\ln 0.9$

Recall that $\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$.
We will take $x = -0.1$ so that $\ln(1+x) = \ln(.9)$
Then $P_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$. Also, $f^{(5)}(x) = 24(1+x)^{-5}$.
Therefore, $R_4 = \frac{24(1+z)^{-5}}{5!}x^5$. In general, $R_n(x) = (-1)^n \frac{(1+z)^{-(n+1)}}{n+1}x^{n+1}$.
When $x = -0.1$, $P_4(x) \approx -0.1 - 0.005 - 0.000333333 - .000025 = -0.105358333$
Since $R_n(x) = \frac{f^{(n+1)(z)}}{(n+1)!}x^{n+1} = (-1)^n \frac{(1+z)^{-(n+1)}}{n+1}x^{n+1}$, where $-0.1 \le z \le 0$.

Since $\ln(1+x)$ is negative and increasing when $-.1 < x < 0$, we need to find n so that $(-1)^n \frac{(1-.1)^{-(n+1)}}{n+1}x^{n+1} <$ 0.000005

When we use $P_4(x)$, our error is at most $\frac{(1-.1)^{-(5)}}{5}(0.1)^5 \approx 0.000084675$.

If we use $P_5(x)$, our error is at most $\frac{(1-.1)^{-(6)}}{6}(0.1)^6 \approx 0.000000314$, so this is a sufficient number of terms to approximate to at least 5 decimal places.

(c) $\sqrt{1.2}$

We will use $f(x) = \sqrt{x}$ centered at $c = 1$ and we will take $x = 1.2$.
Then $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$, $f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$, $f^{(4)}(x) = -\frac{15}{16}x^{-\frac{7}{2}}$, and $f^{(5)}(x) = -\frac{105}{32}x^{-\frac{9}{2}}$.
Then $f(1) = 1$, $f'(1) = \frac{1}{2}$, $f''(1) = -\frac{1}{4}$, $f'''(1) = \frac{3}{8}$, and $f^{(4)(1)=-\frac{15}{16}}$.
Hence $a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = -\frac{1}{8}$, $a_3 = \frac{1}{16}$, and $a_4 = -\frac{5}{128}$
Thus $P_4(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$
and $R_4(x) = \frac{f^{(5)}(z)}{5!}(x-1)^5 = \frac{7z^{-\frac{9}{2}}}{256}(x-1)^5$
Thus $\sqrt{1.2} \approx P_4(1.2) = 1 + \frac{1}{2}(0.2) - \frac{1}{8}(0.2)^2 + \frac{1}{16}(0.2)^3 - \frac{5}{128}(0.2)^4 \approx 1.0954375$
The error of this approximation is at most: $\frac{7(1.2)^{-\frac{9}{2}}}{256}(0.2)^5 \approx .000003852$
Hence this estimate is already sufficient to approximate to 5 decimal places (one can easly verify that $P_3(x)$ is only accurate to 4 decimal places).

5. Find the first four terms of the binomial series for each of the following.

(a) $(1+x)^{\frac{5}{4}}$

$(1+x)^{\frac{5}{4}} \approx 1 + \frac{5}{4}x + \frac{\frac{5}{4}\cdot\frac{1}{4}}{2!}x^2 + \frac{\frac{5}{4}\cdot\frac{1}{4}\cdot\frac{-3}{4}}{3!}x^3 + \cdots$

$= 1 + \frac{5}{4}x + \frac{5}{32}x^2 - \frac{5}{128}x^3 + \cdots$

(b) $(1+4x^3)^{\frac{1}{3}}$

Let $u = 4x^3$ and consider $(1+u)^{\frac{1}{3}}$.

$(1+u)^{\frac{1}{3}} \approx 1 + \frac{1}{3}u + \frac{\frac{1}{3}\cdot\frac{-2}{3}}{2!}u^2 + \frac{\frac{1}{3}\cdot\frac{-2}{3}\cdot\frac{-5}{3}}{3!}u^3 + \cdots$

$= 1 + \frac{1}{3}(4x^3) - \frac{1}{9}(4x^3)^2 + \frac{5}{81}(4x^3)^3 + \cdots$

$= 1 + \frac{4}{3}4x^3 - \frac{16}{9}x^6 + \frac{320}{81}x^9 + \cdots$

(c) $\dfrac{x^2}{(1-x^2)^{\frac{1}{2}}}$

Let $u = -x^2$ and consider $(1+u)^{-\frac{1}{2}}$.

$(1+u)^{-\frac{1}{2}} \approx 1 - \frac{1}{2}u + \frac{\frac{-1}{2}\cdot\frac{-3}{2}}{2!}u^2 + \frac{\frac{-1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}}{3!}u^3 + \cdots$

$= 1 - \frac{1}{2}u + \frac{3}{8}u^2 - \frac{5}{16}u^3 + \cdots$

$= 1 - \frac{1}{2}(-x^2) - \frac{3}{8}(-x^2)^2 + \frac{5}{16}(-x^2)^3 + \cdots = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots$

Then $\dfrac{x^2}{(1-x^2)^{\frac{1}{2}}} = x^2\left(1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots\right) = x^2 + \frac{1}{2}x^4 + \frac{3}{8}x^6 + \frac{5}{16}x^8 + \cdots$

6. Use the binomial series to expand each of the following.

(a) $(1+x)^5$

$(1+x)^5 = 1 + 5x + \frac{5\cdot4}{2!}x^2 + \frac{5\cdot4\cdot3}{3!}x^3 + \frac{5\cdot4\cdot3\cdot2}{4!}x^4 + \frac{5\cdot4\cdot3\cdot2\cdot1}{5!}x^5 + 0 + 0 + \cdots = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$

(b) $(1-4x)^4$

Let $u=-4x$. Then $(1+u)^4 = 1+4u+\frac{4\cdot3}{2!}u^2+\frac{4\cdot3\cdot2}{3!}u^3+\frac{4\cdot3\cdot2\cdot1}{4!}u^4+0+0+\cdots = 1+4u+6u^2+4u^3+u^4$

Therefore, $(1-4x)^4 = 1+4(-4x)+6(-4x)^2+4(-4x)^3+(-4x)^4 = 1-16x+96x^2-256x^3+256x^4$

(c) $(1-2x^5)^3$

Let $u=-2x^5$. Then $(1+u)^3 = 1+3u+\frac{3\cdot2}{2!}u^2+\frac{3\cdot2\cdot1}{3!}u^3+0+0+\cdots = 1+3u+3u^2+u^3$

Therefore, $(1-2x^5)^3 = 1+3(-2x^5)+3(-2x^5)^2+(-2x^5)^3 = 1-6x^5+12x^{10}-8x^{15}$

7. Use a series to approximate $\displaystyle\int_0^1 \frac{1-\cos x}{x^2}\,dx$ to within 5 decimal places of accuracy.

Recall that $\cos^x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots$.

Then $1-\cos x = \frac{x^2}{2}-\frac{x^4}{4!}+\frac{x^6}{6!}-\frac{x^8}{8!}+\cdots+(-1)^{n+1}\frac{x^{2n}}{(2n)!}+\cdots$

Therefore, $\dfrac{1-\cos x}{x^2} = \frac{1}{2}-\frac{x^2}{4!}+\frac{x^4}{6!}-\frac{x^6}{8!}+\cdots+(-1)^{n+1}\frac{x^{2n-2}}{(2n)!}+\cdots$

From this, we evaluate $\displaystyle\int_0^1 \frac{1-\cos x}{x^2}\,dx = \int_0^1 \frac{1}{2}-\frac{x^2}{4!}+\frac{x^4}{6!}-\frac{x^6}{8!}+\cdots+\cdots\,dx = \frac{1}{2}-\frac{x^3}{3\cdot4!}+\frac{x^5}{5\cdot6!}-\frac{x^7}{7\cdot8!}+\cdots\Big|_0^1$

$= \frac{1}{2}-\frac{1}{3\cdot4!}+\frac{1}{5\cdot6!}-\frac{1}{7\cdot8!}+\cdots$

Notice that this is an alternating series whose terms are both decreasing (since the demonimators are increasing) and approach zero. Therefore, we may apply the Altermnating Series Test Error Theorem. Since $\frac{1}{7\cdot8!}\approx 0.000003543 < 0.000005$, then adding the preceding terms will yield an approximation of the original that is good to 5 decimal places.

Hence $\displaystyle\int_0^1 \frac{1-\cos x}{x^2}\,dx \approx \frac{1}{2}-\frac{1}{3\cdot4!}+\frac{1}{5\cdot6!}\approx 0.48639.$

8. Use series to evaluate the limit $\displaystyle\lim_{x\to\infty}\frac{e^x-e^{-x}}{x}$

Sorry. This was a typo. The problem should be: Use series to evaluate the limit $\displaystyle\lim_{x\to0}\frac{e^x-e^{-x}}{x}$

Recall that $e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$.

Then $e^{-x} = 1-x+\frac{x^2}{2}-\frac{x^3}{3!}+\cdots+(-1)^(n)\frac{x^n}{n!}+\cdots$.

Therefore, $e^x-e^{-x} = 0+2x+0+2\cdot\frac{x^3}{6}+0+2\cdot\frac{x^5}{5!}\cdots$

Hence $\displaystyle\lim_{x\to0}\frac{e^x-e^{-x}}{x} = \lim_{x\to0}\frac{2x+\frac{x^3}{3}+\frac{x^5}{60}\cdots}{x}$

$\displaystyle= \lim_{x\to0}\frac{2+\frac{x^2}{3}+\frac{x^4}{60}\cdots}{1} = \frac{2+0+0}{1} = 2$

9. Express the following polar equations in rectangular coordinates:

(a) $r = -5\cos\theta$

Multiplying both sides by r gives $r^2 = -5r\cos\theta$.
Then $x^2 + y^2 = -5x$, or $x^2 + 5x + y^2 = 0$
Completing the square, $(x + \frac{5}{2})^2 + y^2 = \frac{5}{4}$

(b) $r = \sin(2\theta)$

By the double angle identity for $\sin\theta$, $r = 2\sin\theta\cos\theta$.
Then $r^3 = 2r\sin\theta r\cos\theta$, or $(x^2 + y^2)^{\frac{3}{2}} = 2xy$
Then $(x^2 + y^2)^3 = 4x^2y^2$

10. Express the following rectangular equations in polar coordinates:

(a) $xy = 1$

$r\cos\theta r\sin\theta = 1$, or $r^2 = \frac{1}{\sin\theta\cos\theta}$
Then $r^2 = \sec\theta\csc\theta$

(b) $x^2 - y^2 = 1$

$r^2\cos^2\theta - r^2\sin^2\theta = 1$, or $r^2(\cos^2\theta - \sin^2\theta) = 1$
By the double angle identity for $\cos\theta$, $r^2\cos(2\theta) = 1$.
Then $r^2 = \frac{1}{\cos(2\theta)}$, or $r^2 = \sec(2\theta)$.

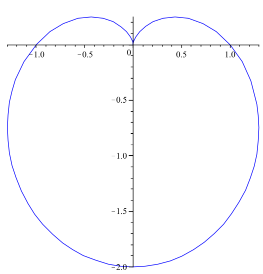
11. Find the equation for a circle with center $(0, -4)$ and passing through the origin in both rectangular and polar coordinates.

Rectangular: $x^2 + (y + 4)^2 = 16$

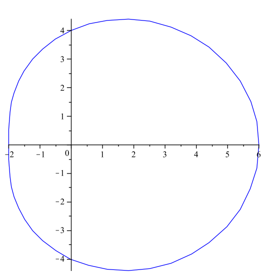
Expanding, $x^2 + y^2 + 8y + 16 = 16$, or $x^2 + y^2 = -8y$
Then $r^2 = -8r\sin\theta$, or $r = -8\sin\theta$

12. Graph each of the following polar equations:

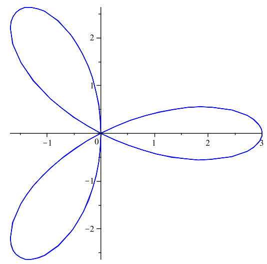
(a) $r = 1 - \sin\theta$



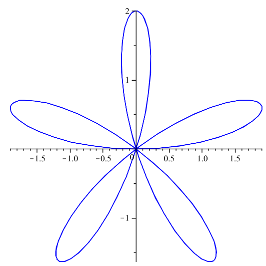
(b) $r = 4 + 2\cos\theta$



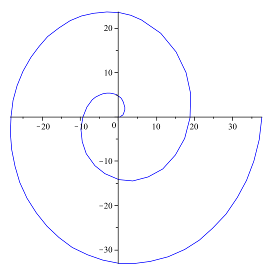
(c) $r = 3\cos(3\theta)$



(d) $r = 2\sin(5\theta)$

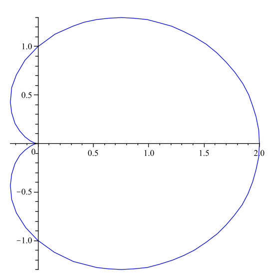


(e) $r = 3\theta$



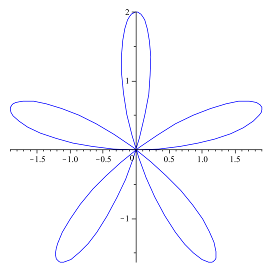
13. Find the area of each of the following polar regions:

(a) the region bounded by the polar graph $r = 1 + \cos\theta$



$$A = 2\int_0^\pi \frac{1}{2}(1+\cos\theta)^2\ d\theta = \int_0^\pi 1 + 2\cos\theta + \cos^2\theta\ d\theta$$
$$= \int_0^\pi 1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos(2\theta)\ d\theta = \frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin(2\theta)\Big|_0^\pi = \frac{3\pi}{2}$$

(b) the region bounded by one loop of the curve $r = 2\sin(5\theta)$

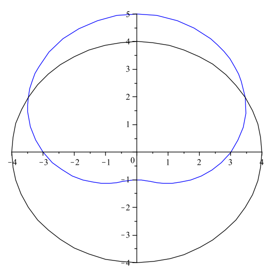


Notice that $2\sin(5\theta) = 0$ when $5\theta = \pi k$ or when $\theta = \frac{\pi k}{5}$. Then the first loop is traced on when $0 \leq \theta \leq \frac{\pi}{5}$.

Therefore, $A = \displaystyle\int_0^{\frac{\pi}{5}} \frac{1}{2}(2\sin(5\theta))^2 \; d\theta = \int_0^{\frac{\pi}{5}} 2\sin^2(5\theta) \; d\theta$

$= 2\displaystyle\int_0^{\frac{\pi}{5}} \frac{1}{2} - \frac{1}{2}\cos(10\theta) \; d\theta = 2\left[\frac{1}{2}\theta - \frac{1}{20}\sin(10\theta)\right]\Bigg|_0^{\frac{\pi}{5}} = \frac{\pi}{5}$

(c) the region inside $r = 3 + 2\sin\theta$ and outside $r = 4$



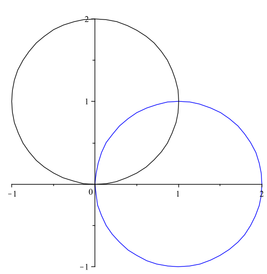
First notice that $3 + 2\sin\theta = 4$ when $2\sin\theta = 1$, or $\sin\theta = \frac{1}{2}$. That is, when $\theta = \frac{\pi}{6} + 2k\pi$ or $\frac{5\pi}{6} + 2k\pi$.

Then, using symmetry, $A = 2\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}((3 + 2\sin\theta)^2 - (4)^2) \; d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9 + 12\sin\theta + 4\sin^2 - 16 \; d\theta$

$= \displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} -7 + 12\sin\theta + 4(\frac{1}{2} - \frac{1}{2}\cos(2\theta)) \; d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} -5 + 12\sin\theta - 2\cos(2\theta) \; d\theta$

$= -5\theta - 12\cos\theta - \sin(2\theta)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \left(-\frac{5\pi}{2}\right) - \left(-\frac{5\pi}{6} - (12)\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = \frac{13\sqrt{3}}{2} - \frac{5\pi}{3}$

(d) the region inside both $r = 2\cos\theta$ and $r = 2\sin\theta$



Notice that $2\cos\theta = 2\sin\theta$ when $\tan\theta = 1$, or when $\theta = \frac{\pi}{4} + k\pi$. Also notice that $2\sin\theta$ is the boundary curve for $0 \leq \theta \leq \frac{\pi}{4}$ while $2\cos\theta$ is the boundary curve for $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$.

Then $A = \displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{2}(2\sin\theta)^2 \; d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}(2\cos\theta)^2 \; d\theta = \int_0^{\frac{\pi}{4}} 2\sin^2\theta \; d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\cos^2\theta \; d\theta$

$= \displaystyle\int_0^{\frac{\pi}{4}} 1 - \cos(2\theta) \; d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 + \cos(2\theta) \; d\theta = \theta - \frac{1}{2}\sin(2\theta)\Bigg|_0^{\frac{\pi}{4}} + \theta + \frac{1}{2}\sin(2\theta)\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \left[\left(\frac{\pi}{4} - \frac{1}{2}\right) - (0)\right] + \left[\left(\frac{\pi}{2} + 0\right) - \left(\frac{\pi}{4} + \frac{1}{2}\right)\right] = \frac{\pi}{2} - 1$