Math 262: Calculus II
Improper Integrals

Recall: There are two ways that in integral can be improper:

1. The integral can have an infinite endpoint, for example, consider $\int_1^\infty \frac{1}{x^2}\, dx$

   In this case, we attempt to evaluate the improper integral by considering what happens as the upper limit of integration approaches positive infinity:

   $$\int_1^\infty \frac{1}{x^2}\, dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\, dx = \lim_{t\to\infty} \int_1^t x^{-2}\, dx = \lim_{t\to\infty} -x^{-1}\Big|_1^t = \lim_{t\to\infty}\left[-\frac{1}{t} - \frac{-1}{1}\right] = 0 + 1 = 1$$

   Thus this improper integral converges to 1.

2. The integrand can become infinite at some point within the interval we are integrating over. If the infinite value occurs at an endpoint, we only need to consider what happens to the value of the integral as we approach the endpoint. If the infinite endpoint occurs inside the interval, we need to consider what happens to the value of the integral as we approach the value both from above and from below. For example, consider $\int_0^2 \frac{1}{(x-1)^2}\, dx$

   We attempt to evaluate the improper integral by considering:

   $$\lim_{t\to1^-} \int_0^t \frac{1}{(x-1)^2}\, dx + \lim_{t\to1^+} \int_t^2 \frac{1}{(x-1)^2}\, dx = \lim_{t\to1^-} -(x-1)^{-1}\Big|_0^t + \lim_{t\to1^+} -(x-1)^{-1}\Big|_t^2 = \lim_{t\to1^-}\left[-\frac{1}{t-1} - \frac{-1}{-1}\right] + \lim_{t\to1^+}\left[-\frac{1}{1} - \frac{-1}{t-1}\right]$$

   Notice that neither of these limits exists, so the integral diverges.

   [In fact, once we see that the first limit does not exist, we can stop and conclude that the integral diverges.]

Using Comparisons to determine whether an improper integral converges or diverges:

The Comparison Test: Suppose that f and g are continuous functions on $[a,\infty)$ and $0 \le f(x) \le g(x)$ for all x in $[a,\infty)$. Then:

(i) If $\int_a^\infty g(x)\, dx$ converges, then $\int_a^\infty f(x)\, dx$ also converges.

(ii) If $\int_a^\infty f(x)\, dx$ diverges, then $\int_a^\infty g(x)\, dx$ also diverges.

Notes:
• This same idea works for improper integrals on intervals of the form $[a,b)$ and $(a,b]$

• The main idea of the first part of the Comparison Test is to show that an improper integral **converges** by finding a **larger** positive valued function that we know converges on the same interval.

• The main idea of the second part of the Comparison Test is to show that an improper integral **diverges** by finding a **smaller** positive valued function that we know diverges on the same interval.

Examples:

1. Use a comparison to determine whether the integral $\int_2^\infty \frac{3x + \sin^2 x}{x^3+1}\,dx$ converges or diverges.

   First, we decide that we guess that this integral converges [notice that the degree of the numerator is two more than that in the denominator]. Therefore, we will try comparing our integrand to a **larger** function whose related integral converges.

   Notice that since $0 \le \sin^2 x \le 1$, then $\frac{3x+\sin^2 x}{x^3+1} \le \frac{3x+1}{x^3+1} \le \frac{3x+1}{x^3} \le \frac{4x}{x^3} = \frac{4}{x^2}$

   [The important principles at work here are that making the numerator of a fraction *larger* **increases** the value of the fraction, and making the denominator of a fraction *smaller* also **increases** the value of the fraction]

   Next, notice that $\int_2^\infty \frac{4}{x^2}\,dx = \lim_{t\to\infty} 4\int_2^t x^{-2}\,dx = \lim_{t\to\infty} -x^{-1}\Big|_2^t$

   $= \lim_{t\to\infty} 4\left[-\frac{1}{t} + \frac{1}{2}\right] = 2.$

   Therefore, our original integral converges by comparison.

   Note: Although we know, by comparison that the original improper integral converges, we have no idea what value it converges to. All we know is that is it converges to some value I satisfying $0 \le I \le 1$

2. Use a comparison to determine whether the integral $\int_2^\infty \frac{\cos x + 4}{x-1}\,dx$ converges or diverges.

   First, we decide that we guess that this integral diverges [notice that the integrand looks a lot like $\frac{1}{x}$, which is divergent on any interval of the form $[a,\infty)$]. Therefore, we will try comparing our integrand to a **smaller** function whose related integral diverges.

   Notice that since $-1 \le \cos x \le 1$, $3 \le \cos x + 4 \le 5$, so $\frac{\cos x+4}{x-1} \ge \frac{3}{x-1} \ge \frac{3}{x}$

   [The important principles at work here are that making the numerator of a fraction *smaller* **decreases** the value of the fraction, and making the denominator of a fraction *larger* also **decreases** the value of the fraction]

   Next, notice that $\int_2^\infty \frac{3}{x}\,dx = \lim_{t\to\infty} = \lim_{t\to\infty} 3\ln x\Big|_2^t$

   $= \lim_{t\to\infty} 3\ln t - 3\ln 2$, which diverges.

   Therefore, our original integral diverges by comparison.