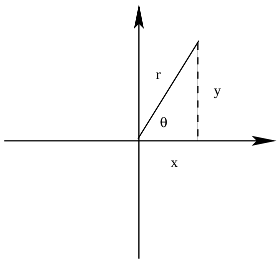
Math 262

Introduction to Polar Coordinates

Recall: In rectangular coordinates on the Cartesian plane, we designate a point by giving its horizontal position relative to a point called the origin using an x-coordinate, and its vertical position relative to the origin using a y-coordinate. Together, these two coordinates uniquely specify a particular point in the plane.

Polar Coordinates are another way of describing points in the Cartesian plane. Here, rather than using horizontal and vertical position, we instead describe the location of a point in the plane relative to a point of origin by giving a distance r and the direction θ. As is customary, the angle is measured in radians, starting at the positive x-axis and moving in the counterclockwise direction. Unlike rectangular coordinates where points can be expressed in exactly one way, a point can be described in multiple ways using polar coordinates since the same direction can be described using many different angles.



Conversion Formulas:
Using standard trigonometric formulas on triangles, we can describe the relationship between rectangular and polar coordinate systems as follows:
$x = r\cos\theta$ and $y = r\sin\theta$
$r = \sqrt{x^2+y^2}$, or $r^2 = x^2+y^2$, and $\tan\theta = \frac{y}{x}$ if $x \neq 0$
These conversion formulas allow us to translate both descriptions of points and equations from rectangular to polar coordinates and vice versa.

Examples:
(a) Given the point $P = (4, -4)$ in rectangular coordinates, $r = \sqrt{x^2+y^2} = \sqrt{16+16} = 4\sqrt{2}$ and $\tan\theta = \frac{y}{x} = \frac{-4}{4} = -1$, and θ is in the 4th quadrant, so $\theta = \tan^{-1}(-1) = -\frac{\pi}{4}$. Therefore, $P = \left(4\sqrt{2}, -\frac{\pi}{4}\right)$
(b) Given the point $Q = (2, \frac{5\pi}{4})$ in polar coordinates, $x = r\cos\theta = 2\cos\frac{5\pi}{4} = (2)\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$. Similarly, $y = r\sin\theta = 2\sin\frac{5\pi}{4} = (2)\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$. Therefore, $Q = (-\sqrt{2}, -\sqrt{2})$
(c) To express $x^2 - y^2 = 1$ in polar coordinates, we substitute using $x = r\cos\theta$ and $y = r\sin\theta$, yielding $(r\cos\theta)^2 - (r\sin\theta)^2 = 1$, or $r^2\cos^2\theta - r^2\sin^2\theta = 1$. Therefore, we have $r^2(\cos^2\theta - \sin^2\theta) = 1$ which, substituting using the using the double angle formula for cos gives $r^2(\cos(2\theta)) = 1$. Thus $r^2 = \frac{1}{\cos(2\theta)}$, or $r = \pm\sqrt{\frac{1}{\cos(2\theta)}}$

Graphing Polar Functions:
To graph polar functions, when possible, we write the polar function in the form $r = f(\theta)$. We then keep track of how increasing θ impacts the magnitude and direction of r. We think of gradually rotating through θ degrees counter-clockwise, and smoothly adjusting r as we go.

Example: $r = 2\cos\theta$

θ	$\cos\theta$	$r = 2\cos\theta$
0	1	2
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3} \approx 1.7$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\sqrt{2} \approx 1.4$
$\frac{\pi}{3}$	$\frac{1}{2}$	1
$\frac{\pi}{2}$	0	0
$\frac{2\pi}{3}$	$-\frac{1}{2}$	-1
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\sqrt{2} \approx -1.4$
$\frac{5\pi}{6}$	$\frac{-\sqrt{3}}{2}$	$-\sqrt{3} \approx -1.7$
π	-1	-2